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Critical Dynamics and Fluctuations for a Mean-Field Model of Cooperative Behavior

Donald A. Dawson[1,2]

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The main objective of this paper is to examine in some detail the dynamics and fluctuations in the critical situation for a simple model exhibiting bistable macroscopic behavior. The model under consideration is a dynamic model of a collection of anharmonic oscillators in a two-well potential together with an attractive mean-field interaction. The system is studied in the limit as the number of oscillators goes to infinity. The limit is described by a nonlinear partial differential equation and the existence of a phase transition for this limiting system is established. The main result deals with the fluctuations at the critical point in the limit as the number of oscillators goes to infinity. It is established that these fluctuations are non-Gaussian and occur at a time scale slower than the noncritical fluctuations. The method used is based on the perturbation theory for Markov processes developed by Papanicolaou, Stroock, and Varadhan adapted to the context of probability-measure-valued processes.

KEY WORDS: Mean field model; cooperative behavior; phase transition; critical fluctuations; universality; probability-measure-valued processes; perturbation theory.

1. INTRODUCTION AND DESCRIPTION OF THE RESULTS

One of the principal problems of stochastic system theory is to describe the behavior of a system which is comprised of a large number of interacting subsystems. In addition an important feature of most systems of this type is a degree of randomness inherent in the microscopic subsystems. The

[1] Department of Mathematics and Statistics, Carleton University, Ottawa, Canada K1S 5B6.

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macroscopic state of the system not only is a manifestation of the statistical behavior of the ensemble of interacting subsystems but also prescribes the environment experienced by the individual subsystems thus creating a feedback control loop. As a consequence of this feedback control mechanism, the random perturbations at the microscopic level are compensated for and the macroscopic system can exhibit a wide range of coherent self-regulating behavior. However in certain special circumstances it is possible for the microscopic fluctuations to become amplified by collective action of the microscopic subsystems and to become important at macroscopic scales. The theoretical study of such "critical" phenomena is complicated by the fact that the distinction between macroscopic and microscopic breaks down in this case and the macroscopic system itself becomes intrinsically stochastic. The main objective of this paper is to investigate this phenomenon in the context of a simple model which can exhibit bistable macroscopic behavior. The model under consideration is a dynamic model of a collection of anharmonic oscillators with a bistable potential together with an attractive mean-field-like interaction.

To describe the system we begin with a single anharmonic oscillator subject to a stochastic disturbance. It is described by the solution of an Itô stochastic differential equation

$$dx(t) = \left[-x^3(t) + x(t) \right] dt + \sigma\, dw(t) \tag{1.1}$$

where $\sigma > 0$ and $\{w(t): t \geqslant 0\}$ is a standard Wiener process. The solution $\{x(t): t \geqslant 0\}$ of Eq. (1.1) is a Markov process with associated Fokker–Planck equation,

$$\partial p(t; x, y)/\partial t = \tfrac{1}{2}\sigma^2\, \partial^2 p(t; x, y)/\partial y^2 - \partial/\partial y\left[(-y^3 + y) p(t; x, y) \right] \tag{1.2}$$

together with the initial condition

$$\lim_{t \downarrow 0} \int f(y) p(t; x, y)\, dy = f(x) \tag{1.3}$$

for every bounded continuous function f. The connection between the process $x(t)$ and the solution of the Fokker–Planck equation $p(t;, \cdot,)$ is given by

$$E\left[f(x(t)) \mid x(0) = x \right] = \int f(y) p(t; x, y)\, dy \tag{1.4}$$

for all bounded continuous functions f where $E[\cdot \mid \cdot]$ denotes the conditional expectation. This one-particle system has been extensively studied beginning with the work of Kramers in 1940 (cf. Dekker,(12) Gardner,(24) Schuss(50)). Although the potential has minima at the points ± 1, the particle can escape from either potential well due to the stochastic fluctuations. In fact the system is ergodic with unique equilibrium distribution given by the probability density function

$$p(x) = Z^{-1}\exp\left[\sigma^{-2}\left(x^2 - \tfrac{1}{2}x^4\right)\right] \tag{1.5}$$

where Z is a normalization constant.

For a system of N independent oscillators of this type, the distribution of oscillators in the two wells will approximate the distribution (1.5) when N is large for all but a small set of exceptional times. However this need not be true if there is an interaction among the oscillators. In this paper we consider the effect of a "mean-field-like" interaction in which each oscillator interacts with every other oscillator. The model is given by the system of Itô stochastic differential equations: for $j = 1, \ldots, N$,

$$dx_j = \left(-x_j^3 + x_j\right)dt + \sigma\, dw_j(t) - \theta\left(x_j - \bar{x}\right)dt \tag{1.6}$$

where $\bar{x}(t) := N^{-1}\sum_{j=1}^{N} x_j(t)$ and $\theta > 0$. The last term in (1.6) can be viewed as an interaction between subsystems which creates a tendency for the subsystems to relax toward the center of gravity of the ensemble. Thus the system provides a simple example of a cooperative interaction.

The system (1.6) has been used to model muscle contraction (cf. Kometani and Shimizu[34]) and similar models have been proposed in chemical kinetics (Horsthemke *et al.*[28]), statistical physics (cf. Haken[26]), and large economic systems (Aoki[2]). Theoretical studies of mean-field-like models have been carried out by Kipnis,[33] and Tanaka and Hitsuda[55].

The solution of the system of N stochastic differential equations (1.6) is a Markov process associated with the Fokker–Planck equation:

$$\begin{aligned}\partial p(t;\mathbf{x},\mathbf{y})/\partial t &= \tfrac{1}{2}\sigma^2 \sum_{j=1}^{N} \partial^2 p(t;\mathbf{x},\mathbf{y})/\partial y_j^2 - \sum_{j=1}^{N} \partial/\partial y_j\left[\left(-y_j^3 + y_j\right)p(t;\mathbf{x},\mathbf{y})\right] \\ &\quad + \theta N^{-1} \sum_{j=1}^{N}\sum_{k\neq j} \partial/\partial y_j\left[(y_j - y_k)p(t;\mathbf{x},\mathbf{y})\right]\end{aligned} \tag{1.7}$$

where $\mathbf{y} = (y_1, \ldots, y_n)$. This system is also ergodic and has a unique invariant (equilibrium) probability measure given by the probability density function

$$p_N(\mathbf{x}) = Z_N^{-1}\exp\left[\beta H_I(x_1, \ldots, x_N)\right] \cdot \prod_{j=1}^{N} \rho(x_j) \tag{1.8}$$

where $\beta = 2/\sigma^2$, and

$$H_I(x_1, \ldots, x_N) = (\theta/2N) \cdot \sum_{j=1}^{N}\sum_{k=1}^{N} x_j x_k$$

$$\rho(x_j) = \exp\left\{\sigma^{-2}\left[(1-\theta)x_j^2 - \tfrac{1}{2}x_j^4\right]\right\}$$

and Z_N is a normalizing constant. The Gibb's distribution (1.8) is the analog of the ϕ^4-Euclidean lattice field on Z^d. In the latter case the interaction energy term $H_I(\cdot)$ is replaced by

$$H(x_1, \ldots, x_N) = \sum_{\mathbf{j}\in\Lambda} \sum_{\mathbf{k}\in\Lambda} J_{\mathbf{jk}} \cdot x_{\mathbf{j}} x_{\mathbf{k}}$$

where Λ is a finite box in Z^d, $J_{\mathbf{jk}}$ is a function of the Euclidean distance $|\mathbf{j}-\mathbf{k}|$ between $\mathbf{j}$ and $\mathbf{k}$, and $\beta > 0$ (the thermodynamic limit involves letting $\Lambda \uparrow Z^d$). If $J_{\mathbf{jk}} \geqslant 0$, the interaction is said to be ferromagnetic. (See Glimm and Jaffe[25] for a complete discussion of this model.) Therefore the equilibrium distribution (1.8) is a ferromagnetic Gibbs distribution. This fact is exploited in our analysis by the use of some standard ferromagnetic inequalities.

In the limit $N \to \infty$, the system (1.6) exhibits a phase transition; this has been analyzed theoretically and demonstrated by numerical simulation in a paper of Desai and Zwanzig.[13] For fixed $\theta > 0$, there exists a critical value σ_c such that the behavior of the system for $\sigma > \sigma_c$ and $\sigma < \sigma_c$ are qualitatively different. For $\sigma > \sigma_c$, the time required for the system to approach equilibrium (with a symmetric distribution) remains bounded in the limit $N \to \infty$. However for $\sigma < \sigma_c$, the time required for the system which is started with a predominance of oscillators in one well to reach equilibrium goes to infinity as $N \to \infty$.

In order to make these statements precise we now formulate the law of large numbers for the system (1.6). When N is large the state of the system can most conveniently be described in terms of the empirical distribution of particles. In other words we consider the empirical process: for each Borel set $A \subset R^1$,

$$X_N(t;A) := N^{-1} \sum_{j=1}^{N} 1_A[x_j(t)] \tag{1.9}$$

where $1_A(\cdot)$ is the indicator function of the set A. In other words, $X_N(t;A)$ is simply the proportion of particles in the set A at time t. Then $X_N(\cdot,\cdot)$ is a Markov process with state space $M_1(R^1)$, the space of probability measures on R^1. The first main result is that for large N, $X_N(t,\cdot)$ evolves in an approximately deterministic fashion and that the evolution is described by a nonlinear partial differential equation.

Result I: The Law of Large Numbers. As $N \to \infty$, $X_N(\cdot,\cdot)$ converges in the sense of weak convergence of measure-valued stochastic processes to the process $\{X_\infty(t) : t \geqslant 0\}$. $X_\infty(t)$ is a deterministic probability-measure-valued process given by the probability density function $p(t;\cdot)$ which satisfies the nonlinear partial differential equation:

$$\partial p(t;\cdot)/\partial t = \tfrac{1}{2}\sigma^2 \partial^2 p(t;x)/\partial x^2 - \partial/\partial x\left\{\left[(1-\theta)x - x^3\right]p(t;x)\right\} - \theta a(t)\cdot \partial p(t;x)/\partial x \tag{1.10}$$

where

$$a(t) = \int x p(t;x)\,dx$$

An equilibrium probability distribution for this deterministic evolution is given by the solution of the pair of equations

$$\begin{gathered}\tfrac{1}{2}\sigma^2 \partial^2 p/\partial x^2 - \partial/\partial x\left\{\left[(1-\theta)x - x^3\right]p\right\} - \theta a\, \partial p/\partial x = 0 \\ a = m(a) = \int x p(x)\,dx\end{gathered} \tag{1.11}$$

Therefore for every solution of the equation $m(a) = a$ there is an equilibrium distribution. Since $m(0) = 0$, there is always one equilibrium distribution with mean zero.

Result II: Existence of a Phase Transition. There exists σ_c, $0 < \sigma_c < \infty$ such that for $\sigma \geqslant \sigma_c$, 0 is the only solution of the equation $m(a) = a$. In this case the unique equilibrium distribution is given by

$$p_0(x) = Z^{-1}\cdot \exp\left\{\sigma^{-2}\left[(1-\theta)x^2 - \tfrac{1}{2}x^4\right]\right\} \tag{1.12}$$

On the other hand for $\sigma < \sigma_c$, there exists nonzero solutions $\pm a_0$ of the equation $m(a) = a$. In this case there exist equilibrium distributions

$$p_{\pm a_0}(x) = Z_{a_0}^{-1}\cdot \exp\left\{\sigma^{-2}\left[(1-\theta)x^2 - \tfrac{1}{2}x^4 \pm 2a_0\theta x\right]\right\} \tag{1.13}$$

This argument for the existence of a phase transition is similar to the classical argument for the existence of a ferromagnetic phase transition using the Curie–Weiss model. However it had been thought that the fluctuations for mean-field-type models are "trivial" in that they always lead to classical fluctuation results with central limit normalization and Gaussian limits. However in an important series of papers Ellis and Newman(16,18,19) have demonstrated that in fact mean-field-type models can exhibit a rich probabilistic structure. Employing ideas from the theory of large deviations they showed that the critical fluctuations of the order parameter can have non-Gaussian limits in the $N \to \infty$ limit. In this paper the study of the critical mean-field-like model is extended to include both critical dynamics and fluctuations in the full empirical probability distribution. This is possible due to the simplicity of this type of model; the analogous questions for ϕ^4-Euclidean lattice fields which have the added complexity of geometrical structure remain open in low dimensions (cf. Aizenmann[1]).

Result III: Fluctuation Theorem for $\sigma > \sigma_c$. According to the law of large numbers, for large N the empirical measure evolves very close to the deterministic evolution given by $X_\infty(t, \cdot)$. In order to investigate the fluctuations from the ideal limiting evolution we consider the fluctuations normalized by the usual central limit scaling:

$$Y_N(t, \cdot) := N^{1/2}\left[X_N(t, \cdot) - X_\infty(t, \cdot)\right] \tag{1.14}$$

The "central limit theorem" for the system (1.6) states that: as $N \to \infty$,

$$Y_N(\cdot, \cdot) \twoheadrightarrow Y(\cdot, \cdot) \tag{1.15}$$

in the sense of weak convergence of probability measures on $C([0, \infty), \mathscr{S}')$ where $\mathscr{S}'$ denotes the space of tempered Schwartz distributions of R^1. The generalized process $Y(\cdot, \cdot)$ is Gaussian and is given by the solution of the linear stochastic evolution equation

$$\partial Y/\partial t = \mathscr{L}_t^* Y + W'(t) \tag{1.16}$$

where $\{W(t) : t \geqslant 0\}$ is a Gaussian Markov process in $\mathscr{S}'$ with covariance:

$$\operatorname{Cov}(\langle W(t), \phi\rangle, \langle W(t), \psi\rangle) = \sigma^2 \int_0^t \int \phi'(x)\psi'(x) X_\infty(s, x)\, dx\, ds \tag{1.17}$$

for every pair ϕ, ψ in $\mathscr{S}$, the space of infinitely differentiable functions which are rapidly decreasing at infinity (cf. Glimm and Jaffe,[25] p. 53). The operator $\mathscr{L}_t^*$ is given by the linearization of the Fokker–Planck operator at the measure $X_\infty(t, \cdot)$, that is,

$$\begin{aligned} \mathscr{L}_t^* Y = {} & \tfrac{1}{2}\sigma^2 \partial^2 Y/\partial x^2 - \partial/\partial x\left\{\left[(1-\theta)x - x^3\right] Y\right\} \\ & - \theta\left[\int y X_\infty(t, y)\, dy\right] \partial Y/\partial x - \theta\langle Y, y\rangle\, \partial X_\infty/\partial x \end{aligned} \tag{1.18}$$

For the equilibrium situation $\mathscr{L}^*$ is defined by (1.18) with $X_\infty(t, \cdot)$ replaced by $p_0(\cdot)$ and $W(t)$ is then a Wiener process in $\mathscr{S}'$. [In the Expression (1.17) for the covariance, $X_\infty(t, \cdot)$ is replaced by $p_0(\cdot)$.] In this case the limit process $Y(t, \cdot)$ is a generalized Ornstein–Uhlenbeck process (cf. Holley and Stroock[27]).

There is an equivalent formulation of (1.15) which brings out the relationship with the classical central limit theorem. For simplicity we state it only for a fixed time $t > 0$. Given $\phi \in \mathscr{S}$, as $N \to \infty$,

$$N^{-1/2} \sum_{j=1}^{N} \left[\phi(x_j(t)) - \int \phi(x) X_\infty(t, x)\, dx\right] \to \langle Y(t), \phi\rangle \qquad \text{(in distribution)} \tag{1.19}$$

where $\langle Y(t), \phi\rangle$ is a Gaussian random variable whose variance can be computed from (1.16) and (1.17) and with mean zero. In particular this implies that the variance of the sum

$$\sum_{j=1}^{N} \phi(x_j(t)) \tag{1.20}$$

grows linearly in N, a property which is characteristic of "weakly dependent" random variables (cf. Cassandro and Jona-Lasinio[6]).

For $\sigma > \sigma_c$ the linearized operator $\mathscr{L}^*$ is stable, that is $\lambda_0 = 0$ is a simple eigenvalue and all other eigenvalues $\lambda_n < 0$. This implies that the linear stochastic evolution equation (1.16) has an equilibrium state which is given by a generalized Gaussian random field which in turn describes the limiting fluctuations in the empirical measure process in the $N \to \infty$ limit. However at $\sigma = \sigma_c$, 0 becomes a double eigenvalue for the linearized operator $\mathscr{L}^*$ and there exists a second eigenfunction $q_0(\cdot)$. This implies that the linear stochastic evolution equation cannot have an equilibrium state and thus cannot serve as an approximation to the equilibrium fluctuations of the original system. In order to obtain a limit theorem which describes the fluctuations in the empirical measure process in the $N \to \infty$ limit at the critical point σ_c, it is necessary to make the following rescaling:

$$U_N(t, \cdot) := N^{1/4}\big[X_N(N^{1/2}t, \cdot) - p_0(x)\,dx\big] \tag{1.21}$$

There are two notable features of this rescaling. The first is that replacement of the factor $N^{1/2}$ in (1.14) by $N^{1/4}$; this implies that the fluctuations are of the order of $N^{-1/4}$ for large N and are thus at a larger scale than noncritical fluctuations which occur at the scale of $N^{-1/2}$. This is equivalent to the statement that the variance of the sum

$$\sum_{j=1}^{N} \phi(x_j(t)) \qquad \text{(in steady state)} \tag{1.22}$$

grows as $N^{3/2}$ rather than linearly in N implying that the random variables $\{x_j(t)\}$ are strongly dependent in equilibrium at the critical point σ_c. The second new feature is that the fluctuation process must be observed in "fast time" $N^{1/2}t$; this is due to the phenomena of "critical slowing down" and means that the fluctuations persist over long time scales.

Result IV: *Critical Fluctuations*. Let $\sigma = \sigma_c$. Then as $N \to \infty$,

$$U_N(\cdot, \cdot) \twoheadrightarrow Z(\cdot, \cdot) \tag{1.23}$$

in the sense of weak convergence of probability measures on $C([0, \infty), M^{\pm}(R^1))$ where

$$Z(t, dx) = z(t) \cdot q_0(x)\,dx \tag{1.24}$$

and $z(t)$ is the solution of the Itô stochastic differential equation

$$dz(t) = -cz^3(t)\,dt + \sigma_* \,dw(t), \qquad c > 0, \quad \sigma_* > 0 \tag{1.25}$$

and $\{w(t) : t \geqslant 0\}$ is a standard one-dimensional Wiener process.

The limit theorem given by (1.23)–(1.25) implies that the critical fluctuations in the empirical process are coherent, that is, the entire empirical distribution process is driven or slaved by the one-dimensional process $z(t)$. This is in sharp contrast to the fluctuations observed in the noncritical case in which the fluctuations are described by a generalized Gaussian random field. This is a manifestation of the "macroscopic" nature of the critical fluctuations.

The signed-measure-valued process $Z(\cdot)$ has an equilibrium distribution, namely,

$$Z_\infty(dx) = \zeta_\infty \cdot q_0(x)\,dx \tag{1.26}$$

where ζ_∞ is a random variable with probability density function

$$p(x) = Z^{-1}\exp\left(-cx^4/2\sigma_*^2\right) \tag{1.27}$$

where Z^{-1} is a normalizing factor. Thus we see that the limiting critical fluctuations for the mean-field-type model (1.6) are non-Gaussian. The distribution (1.27) agrees with that obtained by Ellis and Newman(16,18,19) for the order parameter.

The results described above have been derived for the special model (1.6). However a careful analysis of the proofs reveal that the results remain valid for a wide range of similar models (for example with different self interactions). In order for the Results III and IV to be true three basic conditions must be met:

(i) the existence of a bifurcation point σ_c;

(ii) the linearized operator $\mathscr{L}^*$ is stable for $\sigma > \sigma_c$, and exhibits 0 as a double eigenvalue at σ_c with eigenfunction q_0,

(iii) the quadratic term (in the Taylor expansion of the nonlinear Fokker–Planck equation) stabilizes the critical fluctuations.

It is reasonable to conjecture that conditions (i)–(iii) are valid for a large class of stochastic systems with "mean-field-type" interaction and therefore the limiting critical fluctuations will have the form given by (1.26), (1.27). This observation illustrates the phenomenon of "universality" that is widely discussed in the statistical physics literature (cf. Ma,(38) Lang,(36) Sinai(53)). This idea is a generalization of the central limit theorem and states that the critical fluctuations arising from strongly dependent random systems must fall into one of relatively small family of "universal" probability laws.

Exact statements of the results and the major steps in the proofs are presented in succeeding sections. Two technical proofs which do not contribute to the development of the major themes are relegated to Appendixes A and B. The proof of the main result of this paper (Result IV) is based on the Papanicolaou, Stroock, Varadhan perturbation theory and methodology.[46] A detailed discussion of this method and its application to the appropriate infinite-dimensional diffusion process is presented in Section 4.2.

2. THE N-PARTICLE SYSTEMS AND MEAN-FIELD LIMIT

2.1. The N-Particle Markov Processes

Consider the evolution of N identical interacting subsystems described by the system of Itô stochastic differential equations: for $i = 1, \ldots, N$,

$$\begin{aligned} dx_i(t) &= a_1(x_i(t))\,dt + a_2(\mathbf{x}(t))_i\,dt + \sigma(x_i(t))\,dw_i(t) \\ x_i(0) &= x_{i,0} \in R^1 \end{aligned} \tag{2.1}$$

where

$$a_2(\mathbf{x})_i = N^{-1} \cdot \sum_{j=1}^{N} v(x_i - x_j)$$

The coefficient $a_1(\cdot)$ represents the self-interaction, $a_2(\cdot)$ represents the pair interaction, $\{w_j(t) : t \geqslant 0, j = 1, \ldots, N\}$ is a system of independent standard Wiener processes, and the coefficient $\sigma(\cdot)$ determines the internal noise characteristics of the individual subsystems.

Under the assumptions: for some $0 \leqslant K < \infty$,

$$\begin{aligned} a_1(x_i)^2 + a_2(\mathbf{x})_i^2 + \sigma^2(x_i) &\leqslant K(1 + |\mathbf{x}|^2) \\ |\sigma(x_i) - \sigma(y_i)| + |a_1(x_i) - a_1(y_i)| + |a_2(\mathbf{x})_i - a_2(\mathbf{y})_i| &\leqslant K|\mathbf{x} - \mathbf{y}| \end{aligned} \tag{2.2}$$

the fundamental result of Itô guarantees that the system (2.1) has a unique strong solution (cf. Ikeda and Watanabe[29]).

For the system of anharmonic oscillators (1.6), $\sigma^2(x_i) = \sigma^2$, $v(y) = -\theta y$, $a_1(x_i) = -x_i^3 + x_i$. Note that the coefficient $a_1(\cdot)$ does not satisfy the condition (2.2). However the fact that the system (1.6) also has a unique strong solution is established using a truncation argument and the fact that the solution is nonexplosive (cf. McKean[42] or Ikeda and Watanabe[29]). The solution of (1.6) is characterized as a Markov process as follows.

Theorem 2.1.1. The solution of the system of stochastic differential equations (1.6) is a Markov diffusion process on R^N. The domain $\mathrm{D}(A_N)$ of the infinitesimal generator A_N contains $C_K^2(R^N)$, the space of twice continuously differentiable functions with compact support in R^N. For $f \in C_K^2(R^N)$,

$$A_N f(\mathbf{x}) = \tfrac{1}{2}\sigma^2 \sum_{j=1}^{N} \partial^2 f(\mathbf{x})/\partial x_j^2 + \sum_{j=1}^{N} \left\{ \left[(1-\theta)x_j - x_j^3 \right] + \theta N^{-1} \cdot \sum_{k=1}^{N} x_k \right\} \partial f(\mathbf{x})/\partial x_j \qquad (2.3)$$

Proof. For this and other basic facts concerning stochastic differential equations and Markov diffusion processes, refer to McKean,[42] Ikeda and Watanabe,[29] and Stroock and Varadhan.[54] ∎

In addition the solution of the system (1.6) has a smooth transition probability density function $\{p(t;\mathbf{x},\mathbf{y}) : t > 0,\ \mathbf{x},\mathbf{y} \in R^N\}$ that satisfies the Fokker–Planck equation:

$$\partial p(t;\mathbf{x},\cdot)/\partial t = A_N^* p(t;\mathbf{x},\cdot) \qquad (2.4)$$

where A_N^* is the adjoint of the infinitesimal generator A_N [cf. Eq. (1.7)].

2.2. The Equilibrium Gibbs Distribution

Consider the *energy functional*

$$H(x_1, \ldots, x_N) = H_I(x_1, \ldots, x_N) + H_S(x_1, \ldots, x_N) \qquad (2.6)$$

where

$$H_I(x_1, \ldots, x_N) = (\theta/2N) \cdot \sum_{j=1}^{N} \sum_{k=1}^{N} x_j x_k$$

$$H_S(x_1, \ldots, x_N) = \tfrac{1}{2} \sum_{j=1}^{N} \left[(1-\theta)x_j^2 - \tfrac{1}{2}x_j^4 \right]$$

The *Gibbs distribution* associated with the energy functional $H(\cdot)$ and the inverse temperature $\beta = 2/\sigma^2$ is given by the probability density function

$$p_N(\mathbf{x}) = Z_N^{-1} \cdot \exp\left[\beta H(x_1, \ldots, x_N) \right] \qquad (2.7)$$

where Z_N is a normalizing factor.

Theorem 2.2. The solution of the system (1.8) of stochastic differential equations is a Markov process in R^N with unique invariant probability measure given by the Gibbs distribution (2.7). Furthermore the corresponding stationary stochastic process is ergodic and reversible in the sense that the semigroup $\{T_t : t \geqslant 0\}$ induced on $L^2(R^N; p_N)$ by the Markov process is self-adjoint.

Proof. By direct calculation,

$$A_N^* p_N(\mathbf{x})\,d\mathbf{x} = 0 \tag{2.8}$$

where A_N^* is the adjoint of the infinitesimal generator A_N. This implies (cf. Varadhan[58]) that $p_N(\mathbf{x})\,d\mathbf{x}$ is an invariant measure. The ergodicity of the process and the resulting uniqueness of the invariant probability measure follows from a result of Khasminskii[32] (cf. Varadhan[58]). The self-adjointness of the infinitesimal generator of the semigroup $\{T_t : t \geqslant 0\}$ acting on $L^2(R^N; p_N)$, the space of functions on R^N which are square integrable with respect to the measure p_N, is a consequence of the representation (3.56) (also refer to Fritz[23] for a discussion of reversibility). ■

2.3. Reformulation in Terms of Probability-Measure-Valued Processes

In order to study the limiting behavior of the system (1.8) as the number of oscillators N goes to infinity, we reformulate the N-particle processes in the framework of probability-measure-valued processes. We begin by formulating the mathematical framework for studying probability-measure-valued diffusions.

Let $M_1(R^1)$ denote the set of probability measures on R^1, furnished with the topology of weak convergence. $M_1(R^1)$ serves as the state space for the family of probability-measure-valued processes. Let $\Omega = C([0,\infty), M_1(R^1))$, the space of continuous functions from $[0,\infty)$ into $M_1(R^1)$. We consider the canonical process $X : [0,\infty) \to M_1(R^1)$ defined by $X(t,\omega,A) := \omega(t,A)$ for $\omega \in \Omega$, $t \geqslant 0$ and Borel set $A \subset R^1$. The *distribution* of a probability-measure-valued diffusion process is determined by a mapping $\mu \to P_\mu$ from $M_1(R^1)$ into the space of probability measures on Ω with the *initial condition* $P_\mu[X(0) = \mu] = 1$.

The applicability of the theory of probability-measure-valued processes to the system (1.8) is a consequence of the "exchangeability" of the system. We next define this concept and use it to obtain the basic probability-measure-valued process.

A stochastic system $\{x_j(\cdot) : j = 1, \ldots, N\}$ is said to be *exchangeable* if the probability law of $\{x_j(\cdot) : j = 1, \ldots, N\}$ is identical to that of $\{x_{\pi(j)} : j = 1, \ldots, N\}$ for every permutation π of $1, \ldots, N$.

Lemma 2.3.1. Let μ_N be an exchangeable probability measure on R^N. Let $\{x_j(t) : j = 1, \ldots, N\}$ denote the solution of the system (1.8) with random initial condition given by μ_N. For $t \geqslant 0$, let

$$X_N(t,A) := N^{-1} \cdot \sum_{j=1}^{N} 1_A(x_j(t)) \tag{2.9}$$

where $1_A(\cdot)$ denotes the indicator function of the Borel set $A \subset R^1$. Then $X_N(t, \cdot)$ is a probability-measure-valued Markov diffusion process.

Proof. We first observe that from the exchangeability of the system of Wiener processes $\{w_j(\cdot): j = 1, \ldots, N\}$, the exchangeability assumption made on the initial distribution and the symmetry built into the coefficients, it follows that the system $\{x_j(t): t \geqslant 0, j = 1, \ldots, N\}$ is exchangeable. As a consequence of this (cf. Dawson and Hochberg[11]), $X_N(t, \cdot)$ is a probability-measure-valued process. The Markov property and sample path continuity follow from that of $\{x_j(\cdot): j = 1, \ldots, N\}$. ∎

The probability-measure-valued process $X_N(\cdot, \cdot)$ is known as the N-particle *empirical measure process*. It can also be characterized as the unique solution of a "martingale problem" on Ω. We now briefly describe this alternate formulation.

A *martingale problem* on Ω is described by a pair $(L, \mathsf{D}(L))$, where L is a linear operator defined on the linear subspace $\mathsf{D}(L)$ of $C(M_1(R^1))$. A *solution* is the distribution $\{P_\mu: \mu \in M_1(R^1)\}$ of a probability-measure-valued stochastic process which satisfies the condition

$$\begin{aligned}&\text{for every } \psi \in \mathsf{D}(L),\ \psi(X(t)) - \int_0^t L\psi(X(s))\,ds\\ &\text{is a } P_\mu\text{-martingale for each } \mu \in M_1(R^1).\end{aligned} \tag{2.10}$$

The basic result due to Stroock and Varadhan[54] is that a solution to a martingale problem which is *unique* defines a Markov diffusion process with state space $M_1(R^1)$ (also refer to Ethier and Kurtz[22]).

We next present the martingale problem associated with the empirical measure process and describe the action of the operator G_N on two classes of functions. Let D_0 denote the family of functions on $M_1(R^1)$ of the form $F_{f,\phi}(\mu) = f(\langle \mu, \phi\rangle)$ where $\langle \mu, \phi\rangle := \int \phi(x)\mu(dx)$, $f \in C_b^2(R^1)$, the space of functions on R^1 with bounded continuous second derivatives and $\phi \in C_K^2(R^1)$. For $F_{f,\phi} \in \mathsf{D}_0$, $\mu = N^{-1} \cdot \sum_{j=1}^N \delta_{x_j}$ where δ_{x_j} is a unit mass at x_j, let

$$\begin{aligned}G_N^X F_{f,\phi}(\mu) = {} & \tfrac{1}{2}\sigma^2 f'(\langle \mu, \phi\rangle)\langle \mu, \partial^2\phi/\partial x^2\rangle + f'(\langle \mu, \phi\rangle)\langle \mu, (-x^3 + x)\,\partial\phi/\partial x\rangle\\ & + (\sigma^2/2N) f''(\langle \mu, \phi\rangle)\langle \mu, (\partial\phi/\partial x)^2\rangle\\ & - \theta f'(\langle \mu, \phi\rangle)\Big[\int\int (x - y)(\partial\phi/\partial x)\,\mu(dx)\,\mu(dy)\Big]\end{aligned} \tag{2.11}$$

Let D_1 denote the family of functions on $M_1(R^1)$ of the form

$$F_f(\mu) = \int_{R^n} f(x_1, \ldots, x_n)\mu_n(d\mathbf{x}), \qquad \text{where} \quad \mu_n(d\mathbf{x}) = \mu(dx_1)\ldots\mu(dx_n) \tag{2.12}$$

where $f \in C_0^\infty(R^n)$, the space of C^∞ functions on R^n which vanish at infinity. For $F_f \in \mathsf{D}_1$, let

$$G_N^X F_f(\mu) = \sum_{j=1}^{n} \int_{R^n} \left\{ \tfrac{1}{2}\sigma^2 \partial^2/\partial x_j^2 + \left[(1-\theta)x_j - x_j^3\right]\partial/\partial x_j \right\}$$

$$\times\ f(x_1, \ldots, x_n)\mu_n(d\mathbf{x}) + (\sigma^2/2N)$$

$$\times \sum_{j=1}^{n} \sum_{k \neq j} \int_{R^n} \partial^2/\partial_{x_j}\partial_{x_k}\left[f(x_1, \ldots, x_n)\right]\delta(x_j - x_k)\mu_{n-1}^j(d\mathbf{x})$$

$$+\theta \sum_{j=1}^{n} \int_{R^{n+1}} x_{n+1} \cdot \partial/\partial x_j \left[f(x_1, \ldots, x_n)\right]\mu_{n+1}(d\mathbf{x}) \qquad (2.13)$$

(In μ_{n-1}^j, j denotes the omitted variable.)

Lemma 2.3.2. The martingale problem associated with $(G_N^X, \mathsf{D}_0 \cup \mathsf{D}_1)$ has a unique solution which is given by the distribution of the process $X_N(\cdot, \cdot)$ defined by (2.9).

Proof. The fact that the distribution of $X_N(\cdot, \cdot)$ provides a solution to the martingale problem can be established by a straightforward computation involving Itô's lemma and the observation that

$$\psi(X_N(t)) = g\left[N^{-1} \cdot \sum_{j=1}^{N} \phi(x_j(t))\right] = \psi^*(x_1(t), \ldots, x_N(t)) \quad (2.14)$$

Using (2.3) we obtain

$$A_N\psi^*(x_1, \ldots, x_N)$$

$$= (\sigma^2/2N)\, g'\left[N^{-1} \cdot \sum_{j=1}^{N} \phi(x_j)\right] \cdot \sum_{j=1}^{N} \partial^2\phi(x_j)/\partial x_j^2$$

$$+ (\sigma^2/2N)\, g''\left[N^{-1} \cdot \sum_{j=1}^{N} \phi(x_j)\right] \cdot \sum_{j=1}^{N} \left[\partial\phi(x_j)/\partial x_j\right]^2$$

$$-\theta g'\left[N^{-1} \cdot \sum_{j=1}^{N} \phi(x_j)\right] N^{-2} \cdot \sum_{j=1}^{N} \sum_{k \neq j} (x_j - x_k)\,\partial\phi(x_j)/\partial x_j$$

$$+ g'\left[N^{-1} \cdot \sum_{j=1}^{N} \phi(x_j)\right] \cdot \sum_{j=1}^{N} \left(-x_j^3 + x_j\right)\partial\phi(x_j)/\partial x_j$$

$$= G_N^X \psi\left(N^{-1} \cdot \sum_{j=1}^{N} \delta_{x_j}\right) \qquad (2.15)$$

The fact that $\psi(X_N(t)) - \int_0^t G_N^X \psi(X(s))\, ds$ is a martingale then follows from

Itô's lemma or equivalently, the martingale characterization of the system (1.6).

The uniqueness is first established in the special case $\theta = 0$ using a duality argument (cf. Dawson and Hochberg[11]). Then the uniqueness in the general case is established by using a Cameron–Martin–Girsanov argument (cf. Dawson[8]).

2.4. A Nonlinear Markov Diffusion Process

Before proceeding to the study of the $N \to \infty$ limit we must introduce the notion of a nonlinear Markov diffusion process (in the sense of McKean[40]). Let $C[0, \infty)$ denote the space of functions which are continuous $[0, \infty)$, furnished with the topology of uniform convergence on bounded intervals. Let $\mathscr{F}$ denote the σ-algebra of Borel subsets of $C[0, \infty)$ and let $\{y(t) : t \geqslant 0\}$ denote the canonical process on $C[0, \infty)$. Finally, let $\theta_t : C[0, \infty) \to C[0, \infty)$ denote the shift $\theta_t y(s) := y(t + s)$ for $s, t \geqslant 0$.

A *nonlinear Markov diffusion* process is prescribed by a family of probability measures $\{P_\mu : \mu \in M_1(R^1)\}$ on $(C[0, \infty), \mathscr{F})$ which satisfy the following conditions:

(i) for each $B \in \mathscr{F}$, $P_{\cdot}(B)$ is a Borel measurable function on $M_1(R^1)$;

(ii) for each $\mu \in M_1(R^1)$, $P_\mu(y(0) \in B) = \mu(B)$, and

(iii) for $B \in \mathscr{F}$, $0 \leqslant s \leqslant t$, $P_\mu(\theta_t y \in B \mid y(s) : 0 \leqslant s \leqslant t) = P_\mu(\theta_t y \in B \mid y(t)) = P_{\nu(t)}(B \mid \theta_t y(0))$, where $\nu(t)$ denotes the probability law of $y(t)$. In the terminology of McKean, $y(t)$ is Markov with nonconstant transition mechanism which depends on t only via the distribution of $y(t)$. If we define $T_t(\nu(0)) := \nu(t)$, then $\{T_t : t \geqslant 0\}$ is a nonlinear semigroup in the sense that for $s, t \geqslant 0$,

$$\nu(t + s) = T_{t+s}(\nu(0)) = T_t(\nu(s)) = T_t(T_s(\nu(0))) \tag{2.16}$$

The *infinitesimal generator*, if it exists, is given by

$$A^* \mu := \lim_{t \downarrow 0} (T_t(\mu) - \mu)/t \tag{2.17}$$

The operator A^* can be nonlinear; however, $\nu(t)$ is still a solution of the nonlinear evolution equation:

$$d\nu(t)/dt = A^* \nu(t) \tag{2.18}$$

An important class of nonlinear Markov diffusion processes is given by the Itô stochastic differential equations:

$$dy(t) = a(\nu(t), y(t))\, dt + \sigma(\nu(t), y(t))\, dw(t) \tag{2.19}$$

where $\{w(t) : t \geqslant 0\}$ denotes a standard Wiener process and $\nu(t)$ denotes the probability law of $y(t)$. Under the appropriate Lipschitz conditions on the coefficients $a(\cdot,\cdot)$ and $\sigma(\cdot,\cdot)$ the sequence of successive approximations

$$
\begin{aligned}
y_0(t) &= y(0) \\
y_{n+1}(t) &= y(0) + \int_0^t \sigma(\nu_n(s), y_n(s))\,dw(s) + \int_0^t a(\nu_n(s), y_n(s))\,ds \qquad (2.20) \\
\nu_{n+1}(t) &= \text{probability law of } y_{n+1}(t)
\end{aligned}
$$

converges to the unique strong solution of (2.19) (refer to McKean,[41] Itô and Watanabe,[30] and Belopol'skaja and Daleckii[3] for various versions of these results).

If the solution process has a sufficiently smooth transition density function $p(t; x, y)$, then it is the unique solution of the nonlinear Fokker–Planck equation

$$
\begin{aligned}
\partial p(t; x, y)/\partial t &= A^* p(t; x, y) \\
&= \tfrac{1}{2}\partial^2/\partial y^2\big[\sigma^2(p(t; x, \cdot), y)p(t; x, y)\big] \\
&\quad - \partial/\partial y\big[a(p(t; x, \cdot), y)p(t; x, y)\big] \qquad (2.21)
\end{aligned}
$$

In the general case the forward equation (2.21) is a *weak equation* for the probability measure $p(t; x, dy)$.

The "*mean field*" or "*nonlinear*" *anharmonic oscillator* is given by the nonlinear Itô stochastic differential equation:

$$
\begin{aligned}
dy(t) &= \big[-y^3(t) + y(t)\big]\,dt + \sigma\,dw(t) - \theta\big[y(t) - m_1(t)\big]\,dt \\
m_1(t) &:= E(y(t)).
\end{aligned} \qquad (2.22)
$$

The existence and uniqueness of a strong solution to this equation does not follow immediately from the results referred to above since the coefficients do not satisfy a global Lipschitz condition. However they are established in the Theorem 2.4.1.

The nonlinear Fokker–Planck equation associated with (2.22) is

$$
\begin{aligned}
\partial p(t; \cdot)/\partial t &= \tfrac{1}{2}\sigma^2\,\partial^2 p(t; \cdot)/\partial y^2 - \partial/\partial y\,\big\{\big[(1-\theta)y - y^3\big]p(t; \cdot)\big\} \\
&\quad - \theta\Big[\int y p(t; dy)\Big]\partial p(t; \cdot)/\partial y := A_\infty p \qquad (2.23)
\end{aligned}
$$

Theorem 2.4.1. (a) the nonlinear stochastic differential equation (2.22) has a unique strong solution. The function $m_1(t)$ is C^∞ and bounded on $[0, T]$ for $T < \infty$. (b) The nonlinear weak equation (2.23) has a unique probability-measure-valued solution.

Proof. Refer to Appendix A. ■

Remark 2.4.1. Equation (2.23) can be reformulated in the context of probability-measure-valued martingale problems. For a function $F_f(\cdot)$ in D_1 with $f \in C_0(R^n)$, let

$$GF_f(\mu) = \sum_{j=1}^{n} \int_{R^n} \left\{ \tfrac{1}{2}\sigma^2 \partial^2/\partial x_j^2 + \left[x_j(1-\theta) - x_j^3 \right] \partial/\partial x_j \right\}$$

$$\times f(x_1, \ldots, x_n) \mu_n(d\mathbf{x})$$

$$+ \theta \sum_{j=1}^{n} \int_{R^{n+1}} x_{n+1} \cdot \partial/\partial x_j \left[f(x_1, \ldots, x_n) \right] \mu_{n+1}(d\mathbf{x}) \tag{2.24}$$

The pair (G, D_1) defines a probability-measure-valued martingale problem. Reinterpreting part (b) of Theorem 2.4.1, it follows that this martingale problem has a unique solution, namely, the deterministic probability-measure-valued process

$$Y(t, dx) := p(t; dx) \tag{2.25}$$

where $p(t; dx)$ is the unique probability-measure-valued solution of Eq. (2.23).

2.5. The Mean Field Limit of the Ensemble of Oscillators

Theorem 2.5.1. Let $\{X_N(\cdot,\cdot) : N = 1, 2, \ldots\}$ denote the sequence of probability-measure-valued diffusions defined by (2.9). As $N \to \infty$, $X_N(\cdot,\cdot)$ converges in the sense of weak convergence of probability measures on $C([0,\infty), M_1(R^1))$ to the deterministic process $\{Y(t) : t \geqslant 0\}$.

Proof. Refer to Appendix B. ■

This result can be interpreted as follows. For very large N, the evolution of a single oscillator is approximated by the nonlinear anharmonic oscillator (2.22). In the terminology of McKean,[40] the motion of a "tagged" particle approaches that of a particle which satisfies the nonlinear stochastic differential equation (2.22), that is, the motion of a particle in a "mean field." Furthermore the motions of two or more tagged particles approach independent copies of the one-particle nonlinear motions. The latter property is known as "propagation of chaos"; a statement of this property in the setting of probability-measure-valued processes is given in the following corollary. For another approach to the propagation of chaos, refer to McKean.[41]

Corollary: Propagation of Chaos. Let $\{x_j(t) : t \geqslant 0; j = 1, \ldots, N\}$ denote the N-particle system defined by (1.6). Let $M_{N:k}(t; dx_1, \ldots, dx_k)$ denote the kth moment measure for the random probability measure

$X_N(t, \cdot)$, that is, for Borel sets $A_1, \ldots, A_k$

$$M_{N;k}(t; A_1, \ldots, A_k) := E(X_N(t, A_1) \ldots X_N(t, A_k)) \tag{2.26}$$

Then as $N \to \infty$,

$$M_{N:k}(t; dx_1, \ldots, dx_k) \to \prod_{j=1}^{k} p(t; dx_j) \qquad \text{for each} \quad t \geqslant 0 \tag{2.27}$$

in the sense of weak convergence of probability measures on R^k, where $p(t; dx)$ denotes the probability-measure-valued solution of Eq. (2.23); provided that (2.27) is assumed to be valid for $t = 0$.

Proof. First note that a probability-measure-valued process is deterministic if and only if its moment measures are product measures. Since the limit process $\{Y(t) : t \geqslant 0\}$ is deterministic, its moment measures are product measures as in the left-hand side of (2.27). Hence it suffices to prove that the moment measures of the processes $X_N(t, \cdot)$ converge to those of $Y(t)$. For $f \in C_K(R^k)$, Theorem 2.4.1 implies that for each $t > 0$,

$$\begin{aligned} \lim_{N\to\infty} E_\mu(F_f(X_n(t))) &= \lim_{N\to\infty} \int \cdots \int f(x_1, \ldots, x_k) M_{N:k}(t; dx_1, \ldots, dx_k) \\ &= \int \cdots \int f(x_1, \ldots, x_k) p(t; dx_1) \cdots p(t; dx_k). \end{aligned} \tag{2.28}$$

But (2.28) immediately implies (2.27) and the proof is complete. ■

3. PHASE TRANSITION FOR THE MEAN FIELD LIMIT

3.1. Equilibrium Distributions

Consider the nonlinear Markov process which was obtained in the mean field limit (2.22), (2.23). The equilibrium probability distributions for this model are characterized as follows.

Lemma 3.1.1. Every equilibrium probability distribution for the nonlinear Markov process given by (2.22), (2.23) with $\sigma^2 > 0$, is given by a solution of the functional-integral equations:

$$p_a(x) = Z_a^{-1} \cdot \exp\left\{\sigma^{-2}\left[(1-\theta)x^2 - \tfrac{1}{2}x^4 + 2a\theta x\right]\right\} \tag{3.1}$$

$$a = \int x \cdot p_a(x)\, dx := m(a) \tag{3.2}$$

where Z_a is a normalizing factor.

Proof. In view of Eq. (2.23), an equilibrium probability density for the system (2.22) must satisfy

$$\tfrac{1}{2}\sigma^2\, \partial^2 p_a/\partial x^2 = \partial/\partial x\left\{\left[(1-\theta)x - x^3 + \theta a\right] p_a(x)\right\} \qquad \text{if} \quad \sigma^2 > 0 \tag{3.3}$$

where $a = m_1(a) = \int z p_a(z)\,dz$. Solving (3.3) we obtain

$$p_a(x) = Z_a^{-1} \cdot \exp\left\{\sigma^{-2}\left[(1-\theta)x^2 - \tfrac{1}{2}x^4 + 2a\theta x\right]\right\}$$

where

$$Z_a = \int \exp\left\{\sigma^{-2}\left[(1-\theta)x^2 - \tfrac{1}{2}x^4 + 2a\theta x\right]\right\} dx$$

and the proof is complete. ■

Therefore there is a one-to-one correspondence between equilibrium distributions and solutions of the equation:

$$m(a) = a \tag{3.4}$$

Note that $a = 0$ is always a solution of Eq. (3.4). In the special case $\sigma^2 = 0$, there are exactly three equilibrium probability measures:

$$p_0(\cdot) = \delta_0, \qquad p_1(\cdot) = \delta_1, \qquad p_{-1}(\cdot) = \delta_{-1} \tag{3.5}$$

In this case $p_0(\cdot)$ is unstable and $p_{\pm 1}(\cdot)$ are both asymptotically stable. The question of the existence of more than one equilibrium distribution in the case $\sigma^2 > 0$ is discussed in detail below.

3.2. The Hierarchy of Moment Equations and Generating Functions

Let $\{y(t) : t \geqslant 0\}$ denote the solution of the nonlinear stochastic differential equation (2.22). Then Itô's formula yields

$$\begin{aligned} dy^k(t) &= ky^{k-1}(t)\,dy(t) + \tfrac{1}{2}k(k-1)\sigma^2 y^{k-2}\,dt \\ &= \left[k(1-\theta)y^k(t) - ky^{k+2}(t) + \tfrac{1}{2}k(k-1)\sigma^2 y^{k-2}(t)\right. \\ &\quad \left. + E(y(t))ky^{k-1}(t)\right] dt + \sigma k y^{k-1}(t)\,dw(t) \end{aligned} \tag{3.6}$$

Taking expectations and setting $m_k(t) := E(y^k(t))$, we obtain the *hierarchy of moment equations* for $k = 1, 2, 3, \ldots,$

$$\begin{aligned} dm_k(t)/dt = k\Big[&(1-\theta)m_k(t) - m_{k+2}(t)\tfrac{1}{2}(k-1)\sigma^2 m_{k-2}(t) \\ &+ \theta m_1(t)m_{k-1}(t)\Big] \end{aligned} \tag{3.7}$$

and $m_0 = 1$.

The hierarchy of moment equations (3.7) implies that the moments of an equilibrium probability distribution for the nonlinear Markov process (2.22) satisfy the system of equations

$$m_{k+2} = (1-\theta)m_k + \tfrac{1}{2}(k-1)\sigma^2 m_{k-2} + \theta m_1 m_{k-1}, \qquad k = 1, 2, 3, \ldots \tag{3.8}$$

The system (3.8) allows us to solve for $m_3, m_4, m_5, \ldots$ in terms of the first two moments m_1 and m_2. Thus there remain two unknown parameters which must be determined. For example,

$$m_3 = m_1 \tag{3.9a}$$

$$m_4 = (1-\theta)m_2 + \tfrac{1}{2}\sigma^2 + \theta m_1^2 \tag{3.9b}$$

$$m_5 = (1-\theta)m_1 + \sigma^2 m_1 + \theta m_1 m_2 \tag{3.9c}$$

$$m_6 = \big((1-\theta)^2 + (3/2)\sigma^2\big)m_2 + \tfrac{1}{2}\sigma^2(1-\theta) + \theta(2-\theta)m_1^2 \tag{3.9d}$$

Given a random variable X the moment generating function $M_X(\cdot)$ (provided it exists) is given by

$$M_X(\xi) := E\big(\exp(\xi X)\big) = \sum_{n=0}^{\infty} m_n \xi^n / n! \tag{3.10}$$

The *cumulants*, k_n, and cumulant generating function $C_X(\cdot)$ are defined by

$$C_X(\xi) := \ln M_X(\xi) = \sum_{n=1}^{\infty} k_n \xi^n / n! \tag{3.11}$$

For example,

$$k_2 = m_2 - m_1^2 := v, \qquad \text{the variance of } X \tag{3.11a}$$

$$k_3 = m_3 - 3m_1 v - m_1^3 \tag{3.11b}$$

$$k_4 = m_4 - 4m_1 m_3 - 3m_2^2 + 12m_1^2 m_2 - 6m_1^4 \tag{3.11c}$$

Let $p_a(\cdot)$, Z_a be defined as in Lemma 3.3.1 and let

$$p_0(x) = Z_0^{-1}\exp\big\{\sigma^{-2}\big[(1-\theta)x^2 - \tfrac{1}{2}x^4\big]\big\} \qquad \text{if} \quad \sigma^2 > 0 \tag{3.12}$$

$$\Lambda(a) := \ln Z_a \tag{3.13}$$

Let $M_a(\cdot)$, $C_a(\cdot)$ denote the moment generating and cumulant generating functions of the probability distribution $p_a(\cdot)$, respectively. Then,

$$C_0(\xi) = \Lambda(\sigma^2\xi/2\theta) - \Lambda(0) \tag{3.14}$$

$$C_a(\xi) = C_0(\xi + 2a\theta) - C_0(2a\theta) \tag{3.15}$$

$$m(a) = (\sigma^2/2\theta)\cdot d\Lambda(\xi)/d\xi|_{\xi=a} \tag{3.16}$$

3.2. Moment Inequalities

The purpose of this section is to derive some important inequalities for the first four moments of equilibrium distributions for the nonlinear Markov process (2.32). We begin by reviewing some basic inequalities arising in the study of the general moment problem and in the study of ferromagnetic systems.

3.2.1. The Hankel Inequalities. In order that a Hamburger moment problem

$$m_n := \int_{-\infty}^{\infty} x^n \mu(dx), \qquad n = 0, 1, 2, \ldots \tag{3.17}$$

have a solution, it is necessary and sufficient that

$$\Delta_n \geqslant 0, \qquad n = 0, 1, 2, \ldots \tag{3.18}$$

where Δ_n is the nth Hankel determinant defined by

$$\Delta_n := \begin{vmatrix} m_0 & m_1 \cdots\cdots & m_n \\ m_1 & m_2 \cdots\cdots & m_{n+1} \\ \cdots & \cdots\cdots & \cdots \\ m_n & m_{n+1} \cdots\cdots & m_{2n} \end{vmatrix} \tag{3.19}$$

Proof. Refer to Shohat and Tamarkin[51]. ■

3.2.2. Ferromagnetic Inequalities. Assume that

$$\rho(dx) = \exp\left[-V(x)\right] dx$$

where $V(\cdot)$ satisfies

$$V(\cdot) \text{ is even, continuous,} \quad \lim_{x \to \pm\infty} V(x) = \infty, \quad \text{and} \tag{3.20}$$

$$V(x) = c_1 + \int_0^x g(y)\, dy \qquad \text{with} \quad g(0) = 0 \quad \text{and} \quad g \text{ convex on } \left[0, \infty\right)$$

For some $K \geqslant 0$, assume that $J_{jj} < K, j = 1, \ldots, N$, $0 \leqslant J_{jk} \leqslant K$ for $1 \leqslant j \neq k \leqslant N$. Assume that for any real h_j, $1 \leqslant j \leqslant N$,

$$Z(h_1, \ldots, h_N) := \int_{R^N} \exp\left(\sum_{j=1}^{N} h_j x_j + \sum_{j,k=1}^{N} J_{jk} x_j x_k \right) \prod_{j=1}^{N} \rho(dx_j) \tag{3.21}$$

where $\rho(dx) = \exp[-V(x)]\,dx$. Then the following inequalities are satisfied:

Griffiths, Hurst, Sherman (GHS):

$$\partial^3/\partial h_{j_1} \partial h_{j_2} \partial h_{j_3}\left[\ln Z(h_1, \ldots, h_N)\right] \leqslant 0 \qquad \text{if} \quad h_j \geqslant 0, \quad 1 \leqslant j \leqslant N \tag{3.22}$$

$$GHS\text{:} \quad d^3/dh^3\left[\ln \int \exp(hx) \cdot \exp(-\gamma x^2) \rho(dx)\right] \leqslant 0 \qquad \text{for all} \quad h \geqslant 0,$$

for all values of γ (including negative values) for which $\exp(-\gamma x^2)\rho(dx)$ is a finite measure. (3.23)

Lebowitz: $\partial^4/\partial h_{j_1} \partial h_{j_2} \partial h_{j_3} \partial h_{j_4}[\ln Z(h_1, \ldots, h_N)] \leqslant 0$

if $h_1 = h_2 = \cdots = h_N = 0$ (3.24)

Proof. Refer to Glimm and Jaffe,[25] Ellis, Monroe and Newman,[15] Ellis and Newman,[17] and Lebowitz.[37] ■

We now apply these general results to obtain a series of inequalities for the equilibrium distributions of the nonlinear Markov process.

Lemma 3.2.1. Let k_3 denote the third cumulant of an equilibrium distribution for the nonlinear Markov system (2.22). (a) If $m_1 \geqslant 0$, then

$$k_3 \leqslant 0 \tag{3.25}$$

(b) If $m_I > 0$, then $v \geqslant (1 - m_I^2)/3$ where $v := m_2 - m_I^2$.

Proof. By (3.2) and (3.15),

$$C_{m_1}(\xi) = C_0(\xi + 2m_1\theta) - C_0(2m_1\theta)$$

Then applying the GHS inequality (3.23) we obtain

$$k_3 = d^3C_{m_1}(\xi)/d\xi^3|_{\xi=0} = d^3C_0(\xi)/d\xi^3|_{\xi=2m_1} \leqslant 0 \qquad \text{if} \quad m_1 \geqslant 0$$

and the proof of (a) is complete.

From (3.9a) we have $m_3 = m_1$. But by (3.11b), $m_3 = k_3 + 3m_1v + m_1^3$. Therefore,

$$m_1 - 3m_1v - m_1^3 = k_3 \leqslant 0.$$

Since we are assuming $m_1 > 0$, this implies that

$$1 - 3v - m_1^2 \leqslant 0$$

and the proof of (b) is complete. ■

Lemma 3.2.2. Let $v(\theta,\sigma) = m_2(\theta,\sigma)$ denote the variance of the equilibrium distribution $p_0(\cdot)$ given by (3.12). (Note that $m_1 = 0$.) (a) For fixed θ and $\epsilon > 0$,

$$\sigma/2^{3/2} - \epsilon \leqslant v(\theta,\sigma) \leqslant \sigma/2^{1/2} + \epsilon \tag{3.26}$$

for all sufficiently large $\sigma > 0$. (b) If $\theta \leqslant 1$, then

$$v(\theta,\sigma) \geqslant \sigma/6^{1/2} \tag{3.27}$$

If $\theta > 1$, then

$$v(\theta,\sigma) \geqslant \sigma^2/2(\theta - 1) - 3\sigma^4/4(\theta - 1)^3 \tag{3.28}$$

Proof. (a) Since $p_0(\cdot)$ is even, $m_1 = m_3 = m_5 = \cdots = 0$. From Eqs. (3.9) we have

$$m_4 = \tfrac{1}{2}\sigma^2 + (1 - \theta)m_2 \tag{3.29}$$

$$m_6 = \left[(1 - \theta)^2 + (3/2)\sigma^2\right]m_2 + \tfrac{1}{2}\sigma^2(1 - \theta) \tag{3.30}$$

By the Hankel inequality $\Delta_2 \geqslant 0$, it follows that

$$m_4 \geqslant m_2^2 \tag{3.31}$$

Hence

$$\tfrac{1}{2}\sigma^2 + (1-\theta)m_2 \geqslant m_2^2$$

Therefore for sufficiently large $\sigma > 0$

$$m_2 \leqslant \sigma/2^{1/2} + \epsilon \tag{3.32}$$

From the Hankel inequality $\Delta_3 \geqslant 0$, we have

$$m_2 m_4 m_6 - m_4^3 + m_2^2 m_4^2 - m_2^3 m_6 \geqslant 0 \tag{3.33}$$

Substituting (3.29) and (3.30) into (3.33) and collecting the highest-order terms in σ we obtain

$$m_2 \geqslant \sigma/2^{3/2} - \epsilon \tag{3.34}$$

for all sufficiently large σ and the proof of (a) is complete.

(b), (c) Using the Schwarz inequality,

$$m_4^2 \leqslant m_2 m_6 \tag{3.35}$$

Hence

$$(1-\theta)^2 m_2^2 + (3/2)\sigma^2 m_2^2 + \tfrac{1}{2}\sigma^2(1-\theta)m_2 \geqslant (1-\theta)^2 m_2^2 + \sigma^2(1-\theta)m_2 + \tfrac{1}{4}\sigma^4$$

Therefore

$$3m_2^2 - (1-\theta)m_2 - \tfrac{1}{2}\sigma^2 \geqslant 0, \quad \text{and}$$

$$m_2 \geqslant (1/6)(1-\theta) + \left((1-\theta)^2 + 6\sigma^2\right)^{1/2}$$

If $\theta \leqslant 1$, this implies that

$$m_2 \geqslant \sigma/6^{1/2}$$

If $\theta > 1$, then using a Taylor expansion we obtain

$$m_2 \geqslant \sigma^2/2(\theta-1) - 3\sigma^4/4(\theta-1)^3$$

and the proof of (b) and (c) is complete. ■

Remark 3.2.1. It is possible to extend the method of Hankel inequalities used in the proof of Lemma 3.2.2.a to obtain successively better estimates of $v(\theta,\sigma)$ by using the inequalities $\Delta_n \geqslant 0$. This can be used to obtain numerical approximations of arbitrarily high accuracy to the true value (cf. Dawson[10]).

3.3. Existence of a Phase Transition for the Mean Field Limit

Let $v(\theta,\sigma)$ denote the variance of the distribution $p_0(x)\,dx$, and $\zeta(\theta,\sigma) := (2\theta/\sigma^2)\cdot v(\theta,\sigma)$. A pair (θ,σ_c) for which $\zeta(\theta,\sigma_c) = 1$ is said to be *critical.*

Theorem 3.3.1. The equation $m(\xi) = \xi$ has a strictly positive solution if and only if

$$\zeta(\theta, \sigma) > 1 \tag{3.36}$$

Proof. From Eq. (3.16),

$$d\Lambda(\xi)/d\xi|_{\xi=a} = 2\sigma^{-2}\theta m(a), \quad \text{and} \tag{3.37}$$

$$d^2\Lambda(\xi)/d\xi^2|_{\xi=a} = (2\theta\sigma^{-2})^2 v(a) = (2\theta\sigma^{-2}) \cdot dm(\xi)/d\xi|_{\xi=a} \tag{3.38}$$

By the GHS inequality (3.23), $d^3\Lambda(\xi)/d\xi^3 \leqslant 0$ if $a \geqslant 0$. Therefore,

$$d^2m(\xi)/d\xi^2|_{\xi=a} = (2\theta\sigma^{-2})^{-1} \cdot d^3\Lambda(\xi)/d\xi^3|_{\xi=a} \leqslant 0, \qquad \text{if} \quad a \geqslant 0 \tag{3.39}$$

This implies that the function $\{m(\xi) : \xi \geqslant 0\}$ is concave and that for $a > 0$,

$$dm(\xi)/d\xi|_{\xi=a} \leqslant dm(\xi)/d\xi|_{\xi=0} = (2\theta\sigma^{-2}) \cdot v(\theta, \sigma) = \zeta(\theta, \sigma) \tag{3.40}$$

Therefore the equation $m(a) = a$ has a strictly positive solution if and only if

$$dm(\xi)/d\xi|_{\xi=0} = \zeta(\theta, \sigma) > 1 \tag{3.41}$$

and the proof is complete. ■

Theorem 3.3.2. Consider the nonlinear Markov process (2.23) for fixed $\theta > 0$.

(a) For all sufficiently large σ, $\zeta(\theta, \sigma) < 1$.

(b) For all sufficiently small positive σ, $\zeta(\theta, \sigma) > 1$.

(c) If $\theta > 0$, then the equation $\zeta(\theta, \sigma) = 1$ has a unique root $0 < \sigma_c < \infty$.

(d) At a critical point (θ, σ_c),

$$m_2 = m_4 = \sigma_c^2/2\theta \tag{3.42}$$

(e) Any critical point (θ, σ_c) must satisfy

$$2^{-1/2} \leqslant (\sigma_c/\theta) \leqslant 2^{1/2} \tag{3.43}$$

Proof. (a) For sufficiently large σ and fixed $\theta > 0$, Lemma 3.2.2.a yields the inequality

$$2\theta\sigma^{-2}v(\theta, \sigma) \leqslant 2\theta\sigma^{-2}(\sigma/2^{1/2} + \epsilon) = 2^{1/2}\theta/\sigma + 2\theta\sigma^{-2}\epsilon < 1 \tag{3.42}$$

The inequality (3.42) together with Theorem 3.3.1 imply that the equation $m(\xi) = \xi$ does not have a strictly positive solution if σ is sufficiently large.

(b) For $\theta \leqslant 1$, Lemma 3.2.2.b implies that for sufficiently small positive σ,

$$2\theta\sigma^{-2}v(\theta, \sigma) \geqslant 2\theta/6^{1/2}\sigma > 1. \tag{3.43}$$

For $\theta > 1$, Lemma 3.2.2.c implies that for sufficiently small positive σ,

$$2\theta\sigma^{-2}v(\theta,\sigma) \geqslant 2\theta\sigma^{-2}\left[\sigma^2/2(\theta-1) - 3\sigma^4/4(\theta-1)^3\right] > 1 \qquad (3.44)$$

Inequalities (3.43) and (3.44) together with Theorem 3.3.1 imply that for fixed $\theta > 0$, the equation $m(\xi) = \xi$ has a strictly positive root a for all sufficiently small σ and the proof of (b) is complete.

(d) This follows immediately from Eq. (3.9.b).

(e) This follows immediately from (3.42) and inequality (3.26).

(c) Differentiating $\zeta(\theta,\sigma) = (2\theta/\sigma^2)\cdot\int x^2 p_0(x)\,dx$ with respect to σ,

$$d\zeta(\theta,\sigma)/d\sigma = -(c^2/\sigma)\left\{\sigma^2 m_2 + (1-\theta)\left[\sigma^2/2 + m_2(1-\theta) - m_2^2\right]\right\} \qquad (3.45)$$

If the equation $\zeta(\theta,\sigma) = 1$ has two or more solutions (in σ), then at one of them we must have

$$d\zeta(\theta,\sigma)/d\sigma \geqslant 0 \qquad (3.46)$$

But at such a point, $m_2 = \sigma^2/2\theta$, and therefore

$$d\zeta(\theta,\sigma)/d\sigma = -(c^2/\sigma)\left[3\sigma^4/4\theta - \sigma^2/2 + \sigma^2/2\theta - \sigma^4/4\theta^2\right] \qquad (3.47)$$

But according to (3.43), at a critical point, $\theta = \sigma/\alpha$ where $2^{-1/2} \leqslant \alpha \leqslant 2^{1/2}$. Then

$$d\zeta(\theta,\sigma)/d\sigma = -(c^2\alpha/4)\cdot\left[3\sigma^2 + 2 - \sigma(2/\alpha + \alpha)\right] \qquad (3.48)$$

But for $2^{-1/2} \leqslant \alpha \leqslant 2^{1/2}$, the right-hand side of (3.47) is strictly negative for all $\sigma > 0$ thus contradicting (3.46). This therefore implies that the equation $\zeta(\theta,\sigma) = 1$ has a unique solution $0 < \sigma_c < \infty$ and the proof of (c) is complete. ■

Remark 3.3.1. The *parabolic cylinder function* $D_\nu(z)$ is defined by (cf. Erdélyi[21])

$$D_\nu(z) := \left[\exp(-\tfrac{1}{4}z^2)/\Gamma(-\nu)\right]\int_0^\infty \exp(-zt - \tfrac{1}{2}t^2)t^{-\nu-1}\,dt, \quad \text{for } \nu < 0 \qquad (3.49)$$

Using this and the definition of $v(\theta,\sigma)$ we obtain

$$v(\theta,\sigma) = \sigma D_{-(3/2)}\left[(\theta-1)/\sigma\right]/2D_{-1/2}\left[(\theta-1)/\sigma\right] \qquad (3.50)$$

Hence the criticality condition becomes

$$D_{-(3/2)}\left[(\theta-1)/\sigma\right]/D_{-1/2}\left[(\theta-1)/\sigma\right] = \sigma/\theta \qquad (3.51)$$

In the special case $\theta = 1$, we obtain

$$\sigma_c = D_{-(3/2)}(0)/D_{-1/2}(0) \simeq 0.956 \qquad (3.52)$$

Desai and Zwanzig[13] have also shown that

$$\exp(\Lambda(a)) = \sigma^{1/2}\pi^{1/2} \sum_{n=0}^{\infty} (n!)^{-1}(a^2\theta^2\sigma^2)^n \exp\left(\tfrac{1}{4}z^2\right) D_{-n-1/2}(z) \quad (3.53)$$

where $z := (\theta - 1)/\sigma$. The critical boundary is plotted in their paper.

3.4. Linearization of the Mean Field Limit

The main objective of this section is to study the linearization of the nonlinear Fokker–Planck equation (2.23) around the equilibrium distribution $p_0(x)\,dx$. We begin by studying the self-adjoint operator associated with the one-particle linear Fokker–Planck equation

$$\partial p(t;x)/\partial t = -\partial/\partial x\left[v(x)p(t;x)\right] + \tfrac{1}{2}\sigma^2 \cdot \partial^2 p(t;x)/\partial x^2 := A^*_{1,a}p(t;x) \quad (3.54)$$

where $v(x) = (1-\theta)x - x^3 + a\theta$. The unique equilibrium distribution for this Markov process is given by

$$p_a(x) = \Psi_0^2 \Big/ \int \Psi_0^2(x)\,dx \quad (3.55)$$

where

$$\Psi_0(x) = \exp\left\{(1/2\sigma^2)\left[(1-\theta)x^2 - \tfrac{1}{2}x^4 + 2a\theta x\right]\right\}$$

Equation (3.54) can be solved by obtaining an eigenfunction expansion (cf. Titchmarsh[57]).

The pair $A^*_{1,a}, A_{1,a}$ can be represented as follows:

$$\begin{aligned} A^*_{1,a}p(x) &= -\Psi_0(x)\cdot B\left[p(x)/\Psi_0(x)\right] \\ A_{1,a}\psi(x) &= -\left[1/\Psi_0(x)\right]\cdot B\left[\psi(x)\Psi_0(x)\right] \end{aligned} \quad (3.56)$$

where

$$B\phi(x) := -\tfrac{1}{2}\sigma^2\,\partial^2\phi(x)/\partial x^2 + V_a(x)\phi(x) \quad (3.57)$$

and the relation between V_a and v is given by the Riccati equation

$$dv(x)/dx + (1/2\sigma^2)v^2(x) + \lambda = V_a(x) \quad (3.58)$$

For the anharmonic oscillator,

$$\begin{aligned} V_a(x) &= \tfrac{1}{2}(1-\theta-3x^2) \\ &\quad + (1/2\sigma^2)\left[(1-\theta)x - x^3\right]^2 \\ &\quad + \lambda + (1/2\sigma^2)\left[a^2\theta^2 + 2a\theta\left((1-\theta)x - x^3\right)\right] \end{aligned} \quad (3.59)$$

where λ is chosen so that the smallest eigenvalue of B is $\lambda_0 = 0$. Since

$V_a(x) \to \infty$ as $|x| \to \infty$, the Schrödinger equation

$$B\Psi_j = \lambda_j \Psi_j \tag{3.60}$$

has a pure point spectrum and there is a basis $\{\Psi_j(x) : j = 0, 1, 2, \ldots, \}$ of $L^2(R^1)$ consisting of eigenfunctions of B and

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \tag{3.61}$$

The eigenfunctions for $A^*_{1,a}$ are $p_n(\cdot)$ defined by

$$p_n(\cdot) := \Psi_0(\cdot)\Psi_n(\cdot), \qquad n = 0, 1, 2, \ldots \tag{3.61}$$

and

$$A^*_{1,a} p_n(x) = -\lambda_n p_n(x) \tag{3.62}$$

The solution of Eq. (3.54) is then

$$p(t; x) = p_0(x) + \Psi_0(x) \sum_{n=1}^{\infty} c_n \Psi_n(x) \exp(-\lambda_n t) \tag{3.63}$$

where the convergence is in $L^2(R^1)$.

The linear operator $A_{1,a}$ can also be associated with a self-adjoint operator on $L^2(p_0)$, the space of functions which are square integrable with respect to the measure $p_0(x)\,dx$, as follows:

$$A^S_{1,a} \zeta_n := -\lambda_n \zeta_n, \qquad \text{where} \quad \zeta_n(x) := \Psi_n(x)/\Psi_0(x) \tag{3.64}$$

and $\{\zeta_n(\cdot) : n = 0, 1, 2, \ldots, \}$ forms an orthonormal basis for $L^2(p_0)$.

We now return to the nonlinear equation:

$$\partial p(t; x)/\partial t = A^*_{1,0} p(t; x) - \theta\left[\int y p(t; y)\,dy\right] \partial p(t; x)/\partial x \tag{3.65}$$

Recall that $p_0(\cdot)$ is a fixed point for the evolution determined by (3.65). The linearization of the Fokker–Planck operator at the equilibrium distribution $p_0(\cdot)$ is given by

$$\begin{aligned} \mathscr{L}^* g(x) := {}& \tfrac{1}{2}\sigma^2 \partial^2 g(x)/\partial x^2 - \partial/\partial x\left\{\left[(1-\theta)x - x^3\right] g(x)\right\} \\ & - \theta\left[\int y g(y)\,dy\right] \cdot \left[\partial p_0(x)/\partial x\right] \end{aligned} \tag{3.66}$$

and the linearized Fokker–Planck equation is

$$\partial p^l(t; x)/\partial t = \mathscr{L}^* p^l(t; x) \tag{3.67}$$

The linearized moment equations are

$$\begin{aligned} dm^l_k(t)/dt &= k\left[(1-\theta)m^l_k(t) - m^l_{k+2}(t) + \tfrac{1}{2}(k-1)\sigma^2 m^l_{k-2}(t)\right. \\ &\qquad \left. + \theta m^l_1(t) m_{k-1}(0)\right] \qquad \text{for } k \text{ odd} \\ dm^l_k(t)/dt &= k\left[(1-\theta)m^l_k(t) - m^l_{k+2}(t)\right. \\ &\qquad \left. + \tfrac{1}{2}(k-1)\sigma^2 m^l_{k-2}(t)\right], \qquad \text{for } k \text{ even} \end{aligned} \tag{3.68}$$

where

$$m_k^l(t) := \int x^k p^l(t;x)\,dx$$

Consider the action of the linearized operator $\mathscr{L}^*$ on a function of the form

$$g(x) = \phi(x)\Psi_0(x)$$

This yields

$$\begin{aligned}\mathscr{L}^* g(x) &= \left\{ -\left[B\phi(x)\right] - \theta\langle\phi, x\Psi_0\rangle \cdot 2\left[(1-\theta)x - x^3\right] \cdot \Psi_0/\sigma^2 Z_0 \right\}\Psi_0 \\ &:= (L^{\#}\phi)\cdot\Psi_0 \end{aligned} \tag{3.69}$$

where B is given by (3.57) with $a = 0$. Note that $L^{\#}\Psi_0 = 0$.

Let $\mathscr{H}_1$ denote the Hilbert subspace of $L^2(R^1)$ spanned by Ψ_0 and $x\Psi_0$. $\mathscr{H}_1^{\perp}$ denotes the orthogonal complement of $\mathscr{H}_1$ and Π the projection on $\mathscr{H}_1^{\perp}$. Then $\Pi L^{\#}\Pi$ is a self-adjoint operator on $\mathscr{H}_1^{\perp}$ and for $\phi, \psi \in \mathscr{H}_1^{\perp}$,

$$(\Pi L^{\#}\Pi\phi, \psi) = -(B\phi, \psi) \tag{3.70}$$

Lemma 3.4.1. Let $\lambda_1^{\#}$ denote the smallest eigenvalue of $-\Pi L^{\#}\Pi$. Then

$$\lambda_1^{\#} \geqslant \lambda_1 > 0.$$

Proof.

$$\begin{aligned}\lambda_1 &= \inf\left\{\langle B\phi, \phi\rangle : \|\phi\| = 1, \phi \in L^2(R^1), \langle\phi, \Psi_0\rangle = 0\right\} \\ &\leqslant \inf\left\{\langle B\phi, \phi\rangle : \|\phi\| = 1, \phi \in L^2(R^1), \langle\phi, \Psi_0\rangle = \langle\phi, x\Psi_0\rangle = 0\right\} \\ &= \lambda_1^{\#}\end{aligned}$$

and the proof is complete. ■

Let $\{T_t^{\#} : t \geqslant 0\}$ denote the semigroup of operators on $\mathscr{H}_1^{\perp}$ generated by $\Pi L^{\#}\Pi$. Note that

$$\int_0^{\infty} \|T_t^{\#}\phi\|\,dt \leqslant \left[\int_0^{\infty} \exp(-\lambda_1^{\#}s)\,ds\right]\|\phi\| = \|\phi\|/\lambda_1^{\#} \tag{3.71}$$

Theorem 3.4.1. Let $\theta > 0$. (a) For $\sigma > \sigma_c$, the linearized system is stable, that is, if

$$p^l(0;x) = \phi(x)\Psi_0(x) \qquad \text{with} \quad \phi \in L^2(R^1) \quad \text{and} \quad \int p^l(0;x)\,dx = 0$$

then

$$m_1^l(t) := \int x p^l(t;x)\,dx \to 0, \quad \text{and} \tag{3.72}$$

$$p^l(t;x)/\Psi_0(x) \to 0 \qquad \text{in } L^2(R^1) \quad \text{as} \quad t \to \infty \tag{3.73}$$

(b) For $\sigma = \sigma_c$, the null space of $\mathscr{L}^*$ is spanned by (p_0, q_0) where $q_0(x) := x \cdot p_0(x)$.

Proof. (a) In view of Lemma 3.4.1, (3.73) follows from (3.72). To prove (3.72) consider the linearized moment equations (3.68). First note that the even moments evolve independently of the odd moments according to the same evolution as the one-particle evolution. In view of the hypothesis on $p(0; \cdot)$, this implies that the even moments m_{2k}^l converge exponentially fast to zero. Now consider the modified linear equations for k odd:

$$d\hat{m}_k^l/dt = k\Big[(1-\theta)\hat{m}_k^l(t) - \hat{m}_{k+2}^l(t) + \tfrac{1}{2}(k-1)\sigma^2\hat{m}_{k-2}^l(t) + \theta a(t) m_{k-1}(0)\Big] \tag{3.74}$$

For $a(t) = a$, $\hat{m}_1^l(t) \to m^l(a)$ as $t \to \infty$, and for $\sigma > \sigma_c$,

$$m^l(a) = a \cdot \big[dm(a)/da\big]\big|_{a=0} < a \qquad \text{by Theorem 3.3.2,} \tag{3.75}$$

where $m(a)$ is defined as in (3.16). Also consider the modified one-particle evolution: for $j = 1, 2$,

$$dx(t) = \big[(1-\theta)x(t) - x^3(t)\big]\,dt + \sigma\,dw(t) + a_j(t)\,dt \tag{3.76}$$

If $a_1(t) \leqslant a_2(t)$, then the corresponding mean processes $m_1^{(j)}(t)$ satisfy

$$m_1^{(1)}(t) \leqslant m_1^{(2)}(t) \tag{3.77}$$

This is proved starting from Eq. (3.76) and using a standard coupling argument; the same property (3.77) is then inherited by the first moment $\hat{m}_1^l(\cdot)$ in the linearized hierarchy (3.74).

On the other hand, from the linear system (3.74) it follows that

$$\hat{m}_1^l(t) = \int_0^t k(t-s)a(s)\,ds + o(t) \tag{3.78}$$

Comparing the case $a(t) \geqslant 0$ with the case $a(t) = 0$, it follows from (3.77) that

$$k(u) \geqslant 0 \tag{3.79}$$

Now let $m_1 > 0$ be chosen and assume that t is sufficiently large so that we can ignore the transient term $o(t)$. Freeze $a(t) = m_1$ for $0 \leqslant t \leqslant t_1$. Then

$$\hat{m}_1^l(t) = \hat{m}_1^l(m_1) + o\big[\exp(-\lambda_1 t)\big] \cdot \big[m_1 - m^l(m_1)\big] \tag{3.80}$$

Therefore we can choose t_1 such that

$$\hat{m}_1^l(t_1) \leqslant \tfrac{1}{2}\big(m_1 + m^l(m_1)\big) = \alpha m_1 \qquad \text{with} \quad 0 < \alpha < 1. \tag{3.81}$$

Now freeze $a(t) = \hat{m}_1^l(t_1)$ for $t_1 \leqslant t \leqslant 2t_1$, and continue this process. Then

$$\lim_{t\to\infty} \hat{m}_1^l(t) = \lim_{n\to\infty} \alpha^n \cdot m_1 = 0 \tag{3.82}$$

But it follows from (3.78) that $a(t) \geqslant \hat{m}_1^l(t) \geqslant 0$; hence $\hat{m}_1^l(t)$ computed above dominates the solution of (3.68), $m_1^l(t)$, thus completing the proof of (3.72).

(b) By (3.55), $p_0(\cdot)$ is the unique solution of $\mathscr{L}^*g = 0$ for which $\int xg(x)\,dx = 0$, $\int g(x)\,dx = 1$. In addition, any two solutions of $\mathscr{L}^*g = 0$ with the same value of $\int xg(x)\,dx$ must differ by a multiple of $p_0(\cdot)$. Thus it suffices to prove that $q_0(x)$ is a special solution of $\mathscr{L}^*g = 0$ with $\int xg(x)\,dx \neq 0$. Now,

$$\begin{aligned}
\mathscr{L}^*q_0 &= x\left(\tfrac{1}{2}\sigma^2\,\partial^2 p_0/\partial x^2 - \partial/\partial x\left\{\left[(1-\theta)x - x^3\right]p_0\right\}\right) \\
&\quad + \tfrac{1}{2}\sigma^2\,\partial p_0/\partial x - \left[(1-\theta)x - x^3\right]p_0 - \theta\left[\int x^2 p_0(x)\,dx\right]\cdot \partial p_0/\partial x \\
&= \left[1 - (2\theta/\sigma^2)\cdot\int x^2 p_0(x)\,dx\right]\left((1-\theta)x - x^3\right)p_0(x) \\
&= \left[1 - \zeta(\theta,\sigma)\right]\left[(1-\theta)x - x^3\right]p_0(x)
\end{aligned} \tag{3.83}$$

The proof of (b) then follows immediately since $\zeta(\theta,\sigma_c) = 1$.

Lemma 3.4.1. The idea of the proof of Theorem 3.4.1.a can be modified to show that $p_0(\cdot)$ is an asymptotically stable fixed point for the full nonlinear Fokker–Planck equation (2.24) at the critical point σ_c. This fact is subsumed by the results of Section 4.2.2.

4. THE INFINITE LIMIT: FLUCTUATION LIMIT THEOREMS

4.1. Fluctuation Theorem for $\sigma > \sigma_c$

In this section we investigate the fluctuations around the infinite limit process which are exhibited by a finite system of size N. First recall that there is actually a qualitative difference in their behaviors since the finite systems are ergodic whereas the infinite system can have more than one invariant probability measure. This underlines the fact that in general it is not valid to interchange the limits $N \to \infty$ and $t \to \infty$.

In the infinite limit mean-field model the empirical distribution is given by identically distributed independent samples from the probability distribution $X^\infty(t, dx) = p(t; x)\,dx$ where $p(t; \cdot)$ is the solution of the nonlinear Fokker–Planck equation (2.23). Let $X_N^\infty(t; \cdot)$ denote the empirical measure corresponding to N identically distributed samples from the distribution $p(t; \cdot)$. Let

$$Y_N^\infty(t; \cdot) := N^{1/2}\cdot\left[X_N^\infty(t; \cdot) - p(t; x)\,dx\right] \tag{4.1}$$

On the other hand we can consider the fluctuations in the N-particle process (2.1). In this case the N particles are not independent. We consider the *empirical fluctuation process*:

$$Y_N(t;\cdot) := N^{1/2}\left[X_N(t;\cdot) - p(t;x)\,dx\right] \tag{4.2}$$

Theorem 4.1.1. (a) Assume that the process $X^\infty(\cdot;\cdot)$ is in steady state, that is, $X^\infty(0;dx) = p_0(x)\,dx$. Then

$$Y_N^\infty(\cdot;\cdot) \to Y^\infty(\cdot;\cdot) \tag{4.3}$$

in the sense of weak convergence of probability measures on $C([0,\infty),\mathscr{S}')$. $Y^\infty(\cdot;\cdot)$ is a generalized random field-valued Gaussian process which is represented as the solution of the linear stochastic evolution equation:

$$\partial Y^\infty/\partial t = A_{1,0}^* Y^\infty + W_0' \tag{4.4}$$

where $A_{1,0}^*$ is defined by (3.54) and $\{W_0(t): t \geqslant 0\}$ is a Gaussian random function of space and time with zero mean and covariance:

$$\operatorname{Cov}\left[\langle W_0(t),\phi\rangle,\langle W_0(s),\psi\rangle\right] = \sigma^2\cdot\min(s,t)\int\phi'(x)\psi'(x)p_0(x)\,dx \tag{4.5}$$

(b) $Y_N(t;\cdot) \twoheadrightarrow Y(\cdot;\cdot)$ in the sense of weak convergence of probability measures on $C([0,\infty),\mathscr{S}')$. $Y(\cdot;\cdot)$ is a generalized random-field-valued Gaussian process which is represented as the solution of the stochastic evolution equation

$$\partial Y/\partial t = \mathscr{L}_t^* Y + W_p' \tag{4.6}$$

where

$$\mathscr{L}_t^* Y = \tfrac{1}{2}\sigma^2\partial^2 Y/\partial x^2 - \partial/\partial x\left\{\left[(1-\theta)x - x^3\right]y\right\}$$
$$-\theta\left[\int yp(t;y)\,dy\right]\partial Y/\partial x - \theta\langle Y,y\rangle\,\partial p(t;x)/\partial x$$

and $\{W_p(t): t \geqslant 0\}$ is a zero mean Gaussian Markov process in $\mathscr{S}'$ with covariance:

$$\operatorname{Cov}(\langle W_p(t),\phi\rangle,\langle W_p(s),\psi\rangle) = \sigma^2\int_0^{\min(s,t)}\int\phi'(x)\psi'(x)p(u;x)\,dx\,du$$

for every pair $\phi,\psi \in \mathscr{S}$. In the equilibrium case, that is, $p(0;x) = p_0(x)$, Y is the solution of the linear stochastic evolution equation:

$$\partial Y/\partial t = \mathscr{L}^* Y + W_0' \tag{4.7}$$

where $\mathscr{L}^*$ is defined by (3.66).

Proof. The proofs of (a) and (b) are similar and for this reason we will only discuss (b). The proof is carried out by first establishing that the processes $Y_N(\cdot;\cdot)$ are weakly compact in the topology of weak convergence of probability measures on $C([0,\infty),S)$ where S is a Hilbert subspace of $\mathscr{S}'$. The proof is completed by showing that any limit point is the solution of a martingale problem associated with an operator $G^{(2)}$ and that this martingale problem has a unique solution which coincides with the solution of the stochastic evolution equation (4.6). The operator $G^{(2)}$ is obtained as the formal limit of the infinitesimal generators $G_N^{(2)}$ of the process pair $\{p(t; dx), Y_N(t; \cdot) : t \geqslant 0\}$.

The details of the proof are lengthy but standard (refer to Holley and Stroock,[27] Dawson and Salehi,[9] Tanaka and Hitsuda[55] for detailed proofs of this type). In this discussion we will omit the proof of weak compactness but will carry out the formal limit process in order to identify the limiting martingale problem.

Let $G_N^{(2)}$ denote the infinitesimal generator of the process pair $\{p(t; dx), Y_N(t; \cdot)\}$ when viewed as a diffusion process with values in a subspace of $\mathscr{S}'(R^1 \times R^1)$. Let D denote the algebra of functions on $\mathscr{S}'(R^1 \times R^1)$ containing those of the form

$$F_f(\Phi, \Psi) := \langle f(x_1, \ldots, x_n; x_{n+1}, \ldots, x_{2n}), \Phi^{\times n} \times \Psi^{\times n} \rangle \tag{4.8}$$

where $\Phi^{\times n}$ denotes the n-fold tensor product of the distribution with itself and $f \in \mathscr{S}(R^{2n})$, the space of C^∞ functions which together with derivatives of all orders are rapidly decreasing at infinity. Then,

$$\begin{aligned}
G_N^{(2)} F_f(\Phi, \Psi) = {} & \sum_{j=1}^{2n} \Big\langle \big\{ \tfrac{1}{2}\sigma^2 \partial^2/\partial x_j^2 + \big[(1-\theta)x_j - x_j^3\big] \partial/\partial x_j \big\} \\
& \quad \times f(x_1, \ldots, x_{2n}), \Phi^{\times n} \times \Psi^{\times n} \Big\rangle \\
& + \theta \sum_{j=1}^{n} \big\langle \big[\partial f(x_1, \ldots, x_{2n})/\partial x_j\big] x_{2n+1}, \Phi^{\times n} \times \Psi^{\times n} \times \Phi^{(2n+1)} \big\rangle \\
& + \theta \sum_{j=n+1}^{2n} \big\langle \big(\partial f(x_1, \ldots, x_{2n})/\partial x_j\big) x_{2n+1}, \Phi^{\times n} \times \Psi^{\times n-1, j} \\
& \quad \times \big[\Phi^{(j)} \times \Psi^{(2n+1)} + \Phi^{(2n+1)} \times \Psi^{(j)}\big] \big\rangle \\
& + \tfrac{1}{2}\sigma^2 \sum_{j=n+1}^{2n} \sum_{\substack{k=n+1 \\ k \neq j}}^{2n} \big\langle \partial^2/\partial x_j \partial x_k \big[f(x_1, \ldots, x_{2n}) \big], \Phi^{\times n} \\
& \quad \times \Psi^{\times n-2, j, k} \times \delta(x_j - x_k) \times \Phi^{(j)} \big\rangle \\
& + N^{-1/2} \Big\{ \theta \sum_{j=1}^{2n} \big\langle \partial/\partial x_j f(x_1, \ldots, x_{2n}) x_{2n+1}, \Phi^{\times n} \times \Psi^{\times n+1} \big\rangle \\
& \quad + \tfrac{1}{2}\sigma^2 \sum_{j=n+1}^{2n} \sum_{\substack{k=n+1 \\ k \neq j}}^{2n} \big\langle \partial^2/\partial x_j \partial x_k f(x_1, \ldots, x_{2n}), \Phi^{\times n} \\
& \quad \times \Psi^{\times n-1, k} \times \delta(x_j - x_k) \big\rangle \Big\}
\end{aligned} \tag{4.9}$$

In $\Phi^{\times n-1,j}$, j refers to the omitted variable and in $\Phi^{(j)}$, j refers to the included variable. [In (5.6) some of the coefficients include the expression $\langle x, \Phi\rangle$, which is not defined for arbitrary $\Phi \in \mathscr{S}'$. However it is possible to verify that the solution process is well-defined on a Hilbert subspace of $\mathscr{S}'$ on which this operation is well defined.]

We now consider the limit of the processes $\{p(t;\cdot), Y_N(t;\cdot)\}$ as $N \to \infty$. We obtain

$$\lim_{N\to\infty} G_N^{(2)} F_f(\Phi, \Psi) = G^{(2)} F_f(\Phi, \Psi) \tag{4.10}$$

where

$$\begin{aligned}
&G^{(2)}F_f(\Phi,\Psi)\\
&\quad= \sum_{j=1}^{2n} \Big\langle \Big\{ \tfrac{1}{2}\sigma^2 \partial^2/\partial x_j^2 + \big[(1-\theta)x_j - x_j^3\big]\partial/\partial x_j \Big\} \\
&\qquad\qquad \times f(x_1, \ldots, x_{2n}), \Phi^{\times n} \times \Psi^{\times n} \Big\rangle \\
&\qquad + \theta \sum_{j=1}^{n} \Big\langle \big[(\partial/\partial x_j) f(x_1, \ldots, x_{2n})\big] x_{2n+1}, \Phi^{\times n} \times \Psi^{\times n} \times \Phi^{(2n+1)} \Big\rangle \\
&\qquad + \theta \sum_{j=n+1}^{2n} \Big\langle (\partial f(x_1, \ldots, x_{2n})/\partial x_j) x_{2n+1}, \Phi^{\times n} \Psi^{\times n-1,j} \\
&\qquad\qquad \times \big[\Phi^{(j)} \times \Psi^{(2n+1)} + \Phi^{(2n+1)} \times \Psi^{(j)}\big] \Big\rangle \\
&\qquad + \tfrac{1}{2}\sigma^2 \sum_{j=n+1}^{2n} \sum_{\substack{k=n+1\\ k\neq j}}^{2n} \Big\langle \partial^2/\partial x_j \partial x_k \big[f(x_1, \ldots, x_{2n})\big], \Phi^{\times n} \\
&\qquad\qquad \times \Psi^{\times n-2,j,k} \times \delta(x_j - x_k) \times \Phi^{(j)} \Big\rangle
\end{aligned} \tag{4.11}$$

The linear operator $G^{(2)}$ defined by (4.11) is identical to that associated with the linear stochastic evolution equation for the pair $\{p(t;\cdot), Y(t;\cdot)\}$ where $p(\cdot;\cdot)$ denotes the solution of the nonlinear Fokker–Planck equation (2.24) and $Y(\cdot;\cdot)$ is the solution of the linear stochastic evolution equation (4.4). Since the martingale problem associated with the operator $G^{(2)}$ has a unique solution, this completes the identification of the limit process.

Remark 4.4.1. For $\sigma > \sigma_c$, the linearized operator $\mathscr{L}^*$ was shown to be stable in Theorem 3.4.1. Using this fact it can be shown that the nonlinear stochastic evolution equation (4.7) has an equilibrium distribution which corresponds to a generalized Gaussian random field. This random field describes the limiting equilibrium fluctuations in the empirical measure process in the $N \to \infty$ limit.

4.2. Critical Fluctuations and Dynamics

4.2.1. Critical Fluctuations in the Order Parameter—The Results of Ellis and Newman. In this section we review the results of Ellis and Newman[16,18,19] concerning the fluctuations in the order parameter at the critical point $\sigma = \sigma_c$.

Recall that the N-particle invariant probability measure is given by

$$p_N(x) = Z_N^{-1}\exp\left[(\theta/N\sigma^2)\left(\sum_{j=1}^{N} x_j\right)^2\right]\prod_{j=1}^{N}\rho(x_j) \tag{4.12}$$

where

$$\rho(x_j) = \exp\left\{\sigma^{-2}\left[(1-\theta)x_j^2 - \tfrac{1}{2}x^4\right]\right\}$$

Now let

$$G(z) := \sigma^2 z^2/4\theta - \Lambda(\sigma^2 z/2\theta) \tag{4.13}$$

where $\Lambda(\cdot)$ is defined as in (3.13). Then using (3.14) we obtain

$$\begin{aligned} &dG(z)/dz|_{z=0} = 0 \\ &d^2G(z)/dz^2|_{z=0} = \sigma^2/2\theta - v(\theta,\sigma) \\ &d^3G(z)/dz^3|_{z=0} = 0 \\ &d^4G(z)/dz^4|_{z=0} := \lambda = -k_4 \end{aligned} \tag{4.14}$$

Note that Lebowitz's inequality (3.24) implies that

$$k_4 = m_4 - 3m_2^2 \leqslant 0 \tag{4.15}$$

(It has been verified by numerical approximation that $k_4 < 0$ for several values of the parameters θ and σ.)

There are three cases to be considered.

Case I. Two-Phase Region. This is the case $\zeta(\theta,\sigma) > 1$, that is, $\sigma^2/2\theta < v(\theta,\sigma)$. In this case $G(\cdot)$ has two local minima.

Case II. One-Phase Region (Noncritical). This is the case $\zeta(\theta,\sigma) < 1$, that is, $\sigma^2/2\theta > v(\theta,\sigma)$. In this case $G(\cdot)$ has a unique global minimum at 0.

Case III. The Critical Case. This is the case $\zeta(\theta,\sigma) = 1$, that is, $\sigma^2/2\theta = v(\theta,\sigma)$. In this case

$$d^2G(z)/dz^2|_{z=0} = 0, \quad \text{and} \tag{4.16}$$

$$G(z) = z^4/4! + o(z^4), \qquad \text{for } z \text{ near zero.} \tag{4.17}$$

Theorem 4.2.1. Let $X_1, \ldots, X_N$ be N random variables distributed according to the probability law $p_N(\cdot)$ given by (4.12) and assume that $\sigma = \sigma_c$ and $\lambda > 0$. Let

$$\bar{X}_N := N^{-1} \sum_{j=1}^{N} X_j$$

(a) The distribution of $\bar{X}_N$ satisfies

$$N^{1/4} \cdot \bar{X}_N + W/N^{1/4} \sim \exp\left[-NG(s/N^{1/4})\right] ds \Big/ \int \exp\left[-NG(s/N^{1/4})\right] ds \qquad (4.18)$$

where W is an $N(0, 1)$-random variable independent of $\bar{X}_N$.

(b) As $N \to \infty$,

$$N^{1/4} \cdot \bar{X}_N \twoheadrightarrow Z^{-1} \exp(-\lambda x^4/4!) \qquad (4.19)$$

in the sense of weak convergence of probability measures on R^1.

Proof. Refer to Ellis and Newman.[18] ■

Remark 4.2.1. The proof of the main result (b) of Theorem 4.2.1 is based on the representation (4.18) in the paper of Ellis and Newman[18] and covers a general class of potentials. However, for the special case we consider, (b) can also be proved using a large deviation theorem due to Richter[48] (this method is used in a paper of Dunlop and Newman[14]).

4.2.2. Formulation of the Main Result on Critical Dynamics. It was noted in Remark 3.4.1 that $p_0(\cdot)$ is an asymptotically stable fixed point for the nonlinear Fokker–Planck equation (2.24) although this property is not inherited by the linearized system at the critical point $\sigma = \sigma_c$. This is a reflection of the dynamical property known as "critical slowing down." This means that at the critical point the dynamics must be "speeded up" in order to observe the full development of the critical fluctuations. Furthermore, in the case of distributed systems critical fluctuations are observed at all scales, a consequence of the fact that the "correlation length" tends to infinity as the critical point is approached.

In order to provide a heuristic description of our results on the critical dynamics consider the one-particle system (3.76) with external input $a_j(t) = a$. The equilibrium probability distribution for this system is

$$p_a(x) = Z_a^{-1} \exp\left\{\sigma^{-2}\left[(1-\theta)x^2 - \tfrac{1}{2}x^4 + 2a\theta x\right]\right\} \qquad (4.20)$$

What the main result below says in heuristic terms is that the empirical distribution for the N-particle system (2.1) at the critical point $\sigma = \sigma_c$ can be approximately described as follows:

$$X_N(N^{1/2}t) \simeq p_{z(t)/N^{(1/4)}}(\cdot) \simeq \left[z(t)/N^{1/4}\right] \partial p_a(\cdot)/\partial a\big|_{a=0} \qquad (4.21)$$

where $\{z(t): t \geqslant 0\}$ is a stationary stochastic process obtained as the solution of a stochastic differential equation (4.25) whose equilibrium probability distribution agrees with that given in (4.19). Note that $X_N(\cdot\,;\,\cdot)$ is observed in "fast time" $N^{1/2}t$; this is necessitated by the critical slowing down. Furthermore, note that the fluctuations in the empirical distribution are coherent, that is, the entire empirical distribution is driven or "slaved" by the process $z(t)$. This is in sharp contrast to the fluctuations observed in the noncritical case in which the fluctuations are described by a generalized Gaussian random field. Again this is a manifestation of the "macroscopic" nature of the fluctuations.

Before stating the main result we describe more precisely the limit process appearing on the right-hand side of (4.21). First note that

$$\partial p_a(x)/\partial a|_{a=0} = 2\theta x p_0(x) \tag{4.22}$$

We define the *signed-measure-valued process* $Z(\cdot\,;\,\cdot)$ as follows:

$$Z(t; dx) := z(t) q_0(x)\, dx \tag{4.23}$$

where

$$q_0(x) := x p_0(x) \tag{4.24}$$

and $\{z(t): t \geqslant 0\}$ is the solution of the stochastic differential equation

$$dz(t) = -cz^3(t)\, dt + \sigma^*\, dw(t) \tag{4.25}$$

where $\sigma^* > 0$ and $c := \lambda\sigma^{*2}/6 = (-k_4)\sigma^{*2}/6$, and $\{w(t): t \geqslant 0\}$ is a standard Wiener process. The state space for $Z(\cdot\,;\,\cdot)$ is $M_q^{\pm} := \{\mu : \mu = zq_0,\ z \in R^1\}$.

We next determine the infinitesimal generator of the process $Z(\cdot\,;\,\cdot)$. For $f \in C^2(R^1)$, $\phi \in L^2(p_0)$, let $F_\phi(\cdot)$ be the function on $M_q^{\pm}$ defined by

$$F_\phi(\mu) := f(\langle\phi, \mu\rangle) = f(\langle\phi, d\mu/dp_0\rangle_{p_0}), \qquad \mu \in M_q^{\pm} \tag{4.26}$$

where

$$\langle\phi, \psi\rangle_{p_0} := \int \phi(x)\psi(x) p_0(x)\, dx$$

Then,

$$G_Z F_\phi(\mu) = -cf'(\langle\phi, \mu\rangle)\langle\phi, q_0\rangle (d\mu/dq_0)^3 + \tfrac{1}{2}\sigma^{*2} f''(\langle\phi, \mu\rangle)\langle\phi, q_0\rangle^2 \tag{4.27}$$

Theorem 4.2.2. Let $X_N(\cdot\,;\,\cdot)$ denote the N-particle systems (2.1) and assume that $\theta > 0$ and that $\sigma = \sigma_c$. Let

$$U_N(t; dx) := N^{1/4}\left[X_N(N^{1/2}t; dx) - p_0(x)\, dx\right] \tag{4.28}$$

If $T < \infty$, and $U_N(0) \twoheadrightarrow a \cdot q_0$ as $N \to \infty$, then

$$U_N(\cdot, \cdot) \twoheadrightarrow Z(\cdot\,;\,\cdot) \tag{4.29}$$

in the sense of weak convergence of probability measures on $C([0, T], M_q^{\pm})$.

Corollary 4.2.2. Let $Z_\infty(dx)$ and $U_N(\infty;\cdot)$ denote the equilibrium states of the processes $Z(\cdot;\cdot)$ and $U_N(\cdot;\cdot)$, respectively. Then

(a) $$U_N(\infty;dx) \twoheadrightarrow Z_\infty(dx) = \zeta_\infty \cdot q_0(x)\,dx \qquad \text{as} \quad N \to \infty$$

in the sense of weak convergence of signed measures on R^1, where ζ_∞ is a random variable with probability density function

$$p(x) = Z^{-1}\exp(-cx^4/2\sigma^{*2}) \tag{4.30}$$

(b) (limiting distribution of the order parameter) as $N \to \infty$,

$$\langle x, U_N(;dx)\rangle = N^{1/4}\cdot \bar{X}_N \twoheadrightarrow v(\theta,\sigma_c)\cdot\zeta_\infty \tag{4.31}$$

in the sense of convergence of probability distributions on R^1.

Proof. The proof is carried out in a series of five steps.

Step I. Generator of U_N and the $N\to\infty$ Limit via Perturbation Theory. Let $\epsilon := N^{-1/4}$ and let G_ϵ denote the generator of $U_N(\cdot;\cdot)$ defined by (4.28). For $F_{f,\phi} \in \mathsf{D}_0$, that is, $F_{f,\phi}(\mu) = f(\langle \mu,\phi\rangle)$ with $f \in C_0^2(R^1)$, $\phi \in C_K^2(R^1)$, the generator of X_N, $G_N^X F_{f,\phi}(\mu)$ is given by (2.11). Since we must consider the fluctuations around the distribution $p_0(\cdot)$, that is, we must consider the "centered" process, we restrict out attention to functions $F_{f,\phi}(\cdot)$ with

$$\int \phi(x)p_0(x)\,dx = 0 \tag{4.32}$$

We denote by D_0^1 the class of functions of this type. For such a function

$$F_{f,\phi}\big[U_N(t)\big] = F_{f,\phi}\big[N^{1/4}X_N(N^{1/2}t)\big] \tag{4.33}$$

Making the appropriate transformations in (2.11) we obtain for $F_{f,\phi}(\cdot) \in \mathsf{D}_0^1$,

$$G_\epsilon F_{f,\phi}(\mu) = \big[\epsilon^{-2}G_1 + \epsilon^{-1}G_2 + G_3 + \epsilon G_4\big]F_{f,\phi}(\mu) \tag{4.34}$$

where

$$\begin{aligned} G_1F_{f,\phi}(\mu) &= f'(\langle \mu,\phi\rangle)\langle \mathscr{L}\phi,\ \mu\rangle \\ G_2F_{f,\phi}(\mu) &= f'(\langle \mu,\phi\rangle)\langle x,\ \mu\rangle\langle \partial\phi/\partial x,\ \mu-(p_0/\epsilon)\rangle \\ G_3F_{f,\phi}(\mu) &= \tfrac{1}{2}\sigma^2 f''(\langle \mu,\phi\rangle)\langle(\partial\phi/\partial x)^2,\ p_0\rangle \\ G_4F_{f,\phi}(\mu) &= \tfrac{1}{2}\sigma^2 f''(\langle \mu,\phi\rangle)\langle(\partial\phi/\partial x)^2,\ \mu-(p_0/\epsilon)\rangle \end{aligned} \tag{4.35}$$

and where $\mathscr{L}$ is the linearized operator:

$$\begin{aligned} \mathscr{L}\phi(x) = {} & \tfrac{1}{2}\sigma^2\,\partial^2\phi(x)/\partial x^2 + \big[(1-\theta)x - x^3\big]\,\partial\phi(x)/\partial x \\ & + \theta x\cdot\int\big[\partial\phi(y)/\partial y\big]p_0(y)\,dy \end{aligned} \tag{4.36}$$

The $\epsilon \to 0$ problem is thus reduced to a problem in perturbation theory. The appropriate methodology for treating this perturbation problem has been developed in an important paper of Papanicolaou, Stroock, and Varadhan.[46] Their results extend the earlier work of Kurtz[35] and Papanicolaou[45] to the setting of martingale problems. The proof of Theorem 4.2.2 is based on their perturbation methodology extended to the context of measure-valued processes.

The core of the analysis and the idea behind the identification of the limit is based on the following formal computation. The idea is to compensate for the singular nature of G_ϵ as $\epsilon \to 0$ by introducing additive renormalization terms. This is done by introducing

$$F^\epsilon := F_{f,\phi} + \epsilon F_1 + \epsilon^2 F_2 \tag{4.37}$$

Then

$$\begin{aligned} G_\epsilon F^\epsilon &= \epsilon^{-2} \cdot G_1 F_{f,\phi} + \epsilon^{-1}\left[G_1 F_1 + G_2 F_{f,\phi}\right] \\ &\quad + \left[G_3 F_{f,\phi} + G_2 F_1 + G_1 F_2 \right] + R(\epsilon) \\ &= G_Z F_{f,\phi} + R(\epsilon), \qquad \text{where} \quad R(\epsilon) \to 0 \quad \text{as} \quad \epsilon \to 0 \end{aligned} \tag{4.38}$$

provided that conditions A, B, C below are satisfied.

Condition A: Let $\mathscr{P}$ denote the projection onto the null space of G_1. Then

$$\mathscr{P} F_{f,\phi} = F_{f,\phi}, \quad \text{that is,} \quad G_1 F_{f,\phi} = 0 \tag{4.39}$$

The heuristic idea behind this condition is that the operator G_1 acts as an "averaging" operator and forces the system to live on its null space. We identify the null space of G_1 in the next step of the proof and verify that $\exp(G_1 t) \to \mathscr{P}$ as $t \to \infty$.

Condition B: To eliminate the ϵ^{-1} term we require that

$$G_1 F_1 + G_2 F_{f,\phi} = 0, \quad \text{that is,} \tag{4.40}$$

$$F_1 = -G_1^{-1} G_2 F_{f,\phi} \tag{4.41}$$

where formally

$$G_1^{-1} v := -\int_0^\infty \exp(G_1 t) v \, dt, \qquad \text{provided that} \quad \mathscr{P} v = 0 \tag{4.42}$$

(In the setting of Papanicolaou,[45] G_1^{-1} plays the role of a "recurrent potential kernel.") In order that (4.41) be well defined we require the "solvability condition"

$$\mathscr{P} G_2 \mathscr{P} = 0. \tag{4.43}$$

This condition is verified in the third step of the proof.

Condition C: In order to determine the appropriate F_2 term we must solve the equation

$$G_1F_2 = G_ZF_{f,\phi} - \left(G_3F_{f,\phi} - G_2G_1^{-1}G_2F_{f,\phi}\right), \quad \text{that is} \tag{4.44}$$

$$F_2 = G_1^{-1}\left[G_ZF_{f,\phi} - \left(G_3F_{f,\phi} - G_2G^{-1}G_2F_{f,\phi}\right)\right] \tag{4.45}$$

In order that (4.45) be well defined we require the "solvability condition":

$$\mathscr{P}\left[G_ZF_{f,\phi} - \left(G_3F_{f,\phi} - G_2G_1^{-1}G_2F_{f,\phi}\right)\right] = 0 \tag{4.46}$$

This solvability condition is automatically satisfied if we define

$$G_ZF_{f,\phi} := \mathscr{P}\left[G_3F_{f,\phi} + G_2F_1\right] \tag{4.47}$$

In order to complete the formal limit calculation it remains to verify that G_Z defined by (4.47) agrees with that defined by (4.27). This verification is also carried out in the third step of the proof.

Step II: *Null Space of the Operator G_1 and G_1^{-1}.*

Lemma 4.2.2.1. The null space of the operator G_1 is spanned by functions of the form $F_{f,1}(\cdot)$ and $F_{f,\phi_0}(\cdot)$, where $\phi_0(x)$ is the odd function of x defined by

$$\begin{aligned}
\phi_0(x) &:= \int_0^x u(x)\,dx \\
u(x) &= (2\theta/\sigma^2)\cdot\exp\left\{-\left[(1-\theta)x^2 - \tfrac{1}{2}x^4\right]/\sigma^2\right\} \\
&\quad\cdot\int_x^\infty \exp\left\{\left[(1-\theta)y^2 - \tfrac{1}{2}y^4\right]/\sigma^2\right\}y\,dy.
\end{aligned} \tag{4.48}$$

Proof. If $F_{f,\phi}(\cdot)$ belongs to the null space of G_1, then

$$G_1F_{f,\phi}(\mu) = f'(\langle\mu,\phi\rangle)\langle\mathscr{L}\phi,\mu\rangle = 0.$$

Therefore we require that $\mathscr{L}\phi = 0$, that is,

$$\begin{aligned}
\mathscr{L}_0\phi &= -\theta x\int(\partial\phi/\partial y)\,p_0(y)\,dy, \qquad \text{where} \\
\mathscr{L}_0\phi &= \tfrac{1}{2}\sigma^2\,\partial^2\phi/\partial x^2 + \left[(1-\theta)x - x^3\right]\partial\phi/\partial x
\end{aligned} \tag{4.49}$$

One solution of (4.49) is $\phi \equiv 1$. To obtain an odd solution consider the equation

$$\tfrac{1}{2}\sigma^2\,\partial^2\phi_0/\partial x^2 + \left[(1-\theta)x - x^3\right]\partial\phi_0/\partial x = -\theta x \tag{4.50}$$

Put $u := \partial\phi_0/\partial x$, an even function. For $0 \leqslant x < \infty$,

$$\partial u/\partial x + (2/\sigma^2)\left[(1-\theta)x - x^3\right]u = -(2\theta x/\sigma^2) \tag{4.51}$$

The only solution of (4.51) for which $\int u(x)p_0(x)\,dx < \infty$ is

$$u(x) = (2\theta/\sigma^2)\cdot\exp\left\{-\left[(1-\theta)x^2 - \tfrac{1}{2}x^4\right]/\sigma^2\right\}$$
$$\cdot\int_x^\infty \exp\left\{\left[(1-\theta)y^2 - \tfrac{1}{2}y^4\right]/\sigma^2\right\} y\,dy \tag{4.52}$$

The function $u(x)$ defined by (4.52) yields a solution of (4.49) provided that $\int u(x)p_0(x)\,dx = 1$. But

$$\int_{-\infty}^\infty u(x)p_0(x)\,dx = (4\theta/\sigma^2)\int_0^\infty\left(\int_x^\infty \exp\left\{\left[(1-\theta)y^2-\tfrac{1}{2}y^4\right]/\sigma^2\right\}y\,dy\right)dx$$
$$= (4\theta/\sigma^2)\int_0^\infty x^2\exp\left\{\left[(1-\theta)x^2-\tfrac{1}{2}x^4\right]/\sigma^2\right\}dx$$
$$= (2\theta/\sigma^2)m_2 = 1$$

since $\sigma = \sigma_c$, and the proof of the lemma is complete. ■

Lemma 4.2.2.2. (a) If $\langle 1, \mu\rangle = \langle\phi_0, \mu\rangle = 0$, then $T_t^*\mu \to 0$ as $t\to\infty$, and $G_1^{*-1} := -\int_0^\infty T_t^*\mu\,dt$ is well defined.

(b) If $\langle\phi, p_0\rangle = \langle\phi, q_0\rangle = 0$, then $T_t\phi\to 0$ as $t\to\infty$, and $G_1^{-1}\phi := -\int_0^\infty T_t\phi\,dt$ is well defined.

Proof. Part (a) is a consequence of (3.71). Part (b) then follows by observing that (a) implies that

$$\langle T_t\phi, \mu\rangle = \langle\phi, T_t^*\mu\rangle \to \langle\phi, ap_0 + bq_0\rangle = 0 \tag{4.53}$$

and the convergence to zero on the right-hand side is exponentially fast.

Remark 4.2.2.1. As a consequence of Lemma 4.2.2.2, it follows that the limiting process as $\epsilon\to 0$ must live on the null space of $\mathcal{L}^*$, that is, the state space for the limit process is the linear space spanned by p_0 and q_0. However, in view of the centering of the process it follows that the state space is actually the one-dimensional linear space spanned by q_0.

Remark 4.2.2.2. As a consequence of Lemma 4.2.2.1 it follows that

$$\mathscr{P}F_{f,\phi} = F_{f,\Pi^*\phi} \tag{4.54}$$

where

$$\Pi^*\phi = \langle\phi, 1\rangle_{L^2(p_0)} + \langle\phi,\phi_0\rangle_{L^2(p_0)}\cdot\phi_0/\|\phi_0\|^2$$

and

$$G_1^{-1}F_{f,\phi}(\mu) = f\left[\left\langle\int_0^\infty T_t(\phi - \Pi^*\phi)\,dt,\ \mu\right\rangle\right]$$

Step III: Identification of the Limit Operator G_Z. The null space of G_1 is spanned by functions of the form

$$F_{f,a+b\phi_0}(\mu) = f(\langle \mu, a + b\phi_0 \rangle) \tag{4.55}$$

In view of the centering of the processes U_N it follows that we are restricting our attention to signed measures μ such that $\langle \mu, 1 \rangle = 0$ and therefore

$$F_{f,a+b\phi_0}(\mu) = f(\langle \mu, b \cdot \phi_0 \rangle) = F_{f,b\phi_0}(\mu) \tag{4.56}$$

Condition A then implies that it suffices to compute $G_Z F_{f,\phi_0}$ for $f \in C_0^2(R^1)$. This means that the limit process, if it exists, must live on the space $M_q^{\pm}$.

The solvability condition required for Condition B is given by (4.43). But this follows immediately since

$$\langle \partial\phi_0/\partial x, q_0 \rangle = 0 \tag{4.57}$$

Lemma 4.2.2.3. Let F^ϵ be defined by (4.37) with $\phi = \phi_0$ and let G_Z be defined as in (4.47). Then

$$G_\epsilon F^\epsilon = G_Z F_{f,\phi_0} + R(\epsilon), \qquad \text{where} \quad R(\epsilon) \to 0 \quad \text{as} \quad \epsilon \to 0 \tag{4.58}$$

Proof.

$$\begin{aligned} G_\epsilon F^\epsilon &= \left(\epsilon^{-2}G_1 + \epsilon^{-1}G_2 + G_3 + \epsilon G_4\right) \cdot \left(F_{f,\phi_0} + \epsilon F_1 + \epsilon^2 F_2\right) \\ &= \epsilon^{-2}G_1 F_{f,\phi_0} + \epsilon^{-1}\left(G_1 F_1 + G_2 F_{f,\phi_0}\right) + \left(G_1 F_2 + G_2 F_1 + G_3 F_{f,\phi_0}\right) \\ &\quad + \epsilon G_3 F_1 + \epsilon^2 G_3 F_2 + \epsilon G_4\left(F_{f,\phi_0} + \epsilon F_1 + \epsilon^2 F_2\right) \end{aligned}$$

The first term is zero since $G_1 F_{f,\phi_0} = 0$. The second term is

$$G_1 F_1 + G_2 F_{f,\phi_0} = -G_1 G_1^{-1} G_2 F_{f,\phi_0} + G_2 F_{f,\phi_0} = 0$$

Finally,

$$\begin{aligned} G_1 F_2 + G_2 F_1 + G_3 F_{f,\phi_0} &= \left(G_Z F_{f,\phi_0} - G_3 F_{f,\phi_0} - G_2 F_1\right) + G_2 F_1 + G_3 F_{f,\phi_0} \\ &= G_Z F_{f,\phi_0} \end{aligned}$$

and the proof is complete. ■

Lemma 4.2.2.4. For $f \in C_0^2(R^1)$, $\mu \in M_q^{\pm}$,

$$\begin{aligned} G_Z F_{f,\phi_0}(\mu) &= (\theta k_4/12) f'(\langle \phi_0, \mu \rangle) \cdot \langle (\phi_0/\langle \phi_0, q_0 \rangle), \mu \rangle^3 \\ &\quad + \tfrac{1}{2}\sigma^2 f''(\langle \phi_0, \mu \rangle)\left[\int (\partial\phi_0/\partial x)^2 p_0(x)\, dx\right] \end{aligned} \tag{4.59}$$

Proof. By (4.47) and (4.41),

$$G_Z F_{f,\phi_0} = \mathscr{P}\left[G_3 F_{f,\phi_0} - G_2 G_1^{-1} G_2 F_{f,\phi_0} \right]$$

But

$$\mathscr{P} G_3 F_{f,\phi_0} = \tfrac{1}{2}\sigma^2 f''(\langle \phi_0, \mu \rangle) \int (\partial \phi_0/\partial x)^2 p_0(x)\, dx \tag{4.60}$$

To determine the first term we begin with

$$G_2 F_{f,\phi_0}(\mu) = \theta f'(\langle \phi_0, \mu \rangle) \cdot \langle x, \mu \rangle \langle \partial\phi_0/\partial x, \mu \rangle$$

Then

$$G_1^{-1} G_2 F_{f,\phi_0}(\mu) = -\theta \int_0^\infty f'(\langle T_t \phi_0, \mu \rangle) \langle T_t x, \mu \rangle \langle T_t(\partial \phi_0/\partial x), \mu \rangle \, dt \tag{4.61}$$

Hence,

$$G_2 G_1^{-1} G_2 F_{f,\phi_0}(\mu)$$

$$\begin{aligned} &= -\theta^2 \langle x, \mu \rangle \int_0^\infty \big[f''(\langle \phi_0, \mu \rangle) \langle \partial \phi_0/\partial x, \mu \rangle \langle T_t x, \mu \rangle \langle T_t(\partial\phi_0/\partial x), \mu \rangle \\ &\qquad + f'(\langle \phi_0, \mu \rangle) \langle \partial/\partial x\, T_t x, \mu \rangle \langle T_t(\partial\phi_0/\partial x), \mu \rangle \\ &\qquad + f'(\langle \phi_0, \mu \rangle) \langle T_t x, \mu \rangle \langle \partial/\partial x\, T_t(\partial\phi_0/\partial x), \mu \rangle \big] \, dt \\ &= -\theta^2 \langle x, \mu \rangle f'(\langle \phi_0, \mu \rangle) \int_0^\infty \langle T_t x, \mu \rangle \langle \partial/\partial x\, T_t(\partial\phi_0/\partial x), \mu \rangle \, dt \end{aligned}$$

Hence,

$$\begin{aligned} \mathscr{P} G_2 G_1^{-1} G_2 F_{f,\phi_0}(\mu) = &-\theta^2 \langle x, \mu \rangle f'(\langle \phi_0, \mu \rangle) \langle x, q_0 \rangle \cdot \langle \phi_0/\langle \phi_0, q_0 \rangle, \mu \rangle^2 \\ &\cdot \int_0^\infty \langle \partial/\partial x\, T_t(\partial\phi_0/\partial x), q_0 \rangle \, dt \end{aligned} \tag{4.62}$$

Note that

$$\begin{aligned} \langle 1, \partial q_0/\partial x \rangle &= \langle 1, p_0\big(1 + (2/\sigma^2)\big[(1-\theta)x^2 - x^4\big]\big) \rangle \\ &= 1 + (2/\sigma^2)\big[(1-\theta)m_2 - m_4\big] = 0 \end{aligned}$$

Therefore,

$$\left\langle \partial/\partial x \left[\int_0^\infty T_t u(x)\, dt \right], q_0 \right\rangle$$

$$= \left\langle \partial/\partial x \left\{ \int_0^\infty T_t \left[u(x) - 2\int_0^\infty u(x) p_0(x)\, dx \right] dt \right\}, q_0 \right\rangle$$

where $u(x) := \partial\phi_0(x)/\partial x$. Let

$$v(x) := \partial V(x)/\partial x, \qquad \text{where}$$
$$V(x) := \int_0^\infty T_t\left[u(x) - 2\int_0^\infty u(x)p_0(x)\,dx\right]dt \tag{4.63}$$

Then $V(x)$ is an even function of x which satisfies the equation

$$-\mathscr{L}_0 V(x) = u(x) - \int_0^\infty u(y)p_0(y)\,dy \tag{4.64}$$

where $\mathscr{L}_0$ is defined as in (4.49). Solving for $v(\cdot)$ using the same method as in (4.51) we obtain

$$\begin{aligned} v(x) = &-(4\theta/\sigma^4)\exp\left\{-\left[(1-\theta)x^2 - \tfrac{1}{2}x^4\right]/\sigma^2\right\} \\ &\left[\int_0^x\int_z^\infty y\cdot\exp\left\{\left[(1-\theta)y^2 - \tfrac{1}{2}y^4\right]/\sigma^2\right\}dy\,dz\right. \\ &-(2/Z_0)\left(\int_0^x \exp\left\{\left[(1-\theta)y^2 - \tfrac{1}{2}y^4\right]/\sigma^2\right\}dy\right) \\ &\left.\times\left(\int_0^\infty\int_z^\infty \exp\left\{\left[(1-\theta)y^2 - \tfrac{1}{2}y^4\right]/\sigma^2\right\}\cdot y\,dy\,dz\right)\right] \end{aligned}$$

Hence

$$\begin{aligned} v(x) = &-(4\theta/\sigma^4)\cdot\exp\left\{-\left[(1-\theta)x^2 - \tfrac{1}{2}x^4\right]/\sigma^2\right\} \\ &\times\left(\int_0^x\int_z^\infty y\cdot\exp\left\{\left[(1-\theta)y^2 - \tfrac{1}{2}y^4\right]/\sigma^2\right\}dy\,dz\right. \\ &\left.-m_2\int_0^x \exp\left\{\left[(1-\theta)y^2 - \tfrac{1}{2}y^4\right]/\sigma^2\right\}dy\right) \end{aligned} \tag{4.65}$$

Then

$$\begin{aligned} &\int_0^\infty v(x)q_0(x)\,dx \\ &= -(4\theta/\sigma^4)\int_0^\infty x\left(\int_0^x\int_z^\infty y\cdot\exp\left\{\left[(1-\theta)y^2 - \tfrac{1}{2}y^4\right]/\sigma^2\right\}dy\,dz\right. \\ &\qquad\left.-m_2\int_0^x \exp\left\{\left[(1-\theta)y^2 - \tfrac{1}{2}y^4\right]/\sigma^2\right\}dy\right)dx \\ &= (4\theta/\sigma^4)\int_0^\infty \tfrac{1}{2}x^2\left(\int_x^\infty y\cdot\exp\left\{\left[(1-\theta)y^2 - \tfrac{1}{2}y^4\right]/\sigma^2\right\}dy\right. \\ &\qquad\left.-m_2\cdot\exp\left\{\left[(1-\theta)x^2 - \tfrac{1}{2}x^4\right]/\sigma^2\right\}\right)dx \end{aligned}$$

In the last step we have used integration by parts and the fact that

$$\lim_{x\to\infty} x^2 \cdot \left[\int_0^x \int_z^\infty y \cdot p_0(y)\,dy\,dz - m_2 \int_0^x p_0(y)\,dy \right]$$
$$= \lim_{x\to\infty} x^2 \cdot \left[\int_0^x y^2 p_0(y)\,dy - m_2 \int_0^x p_0(y)\,dy \right]$$
$$= \lim_{x\to\infty} x^2 \left[\int_x^\infty y^2 p_0(y)\,dy - m_2 \int_x^\infty p_0(y)\,dy \right] = 0$$

since

$$\int_x^\infty p_0(y)\,dy \leqslant x^{-3} \cdot \exp(-x^4), \quad \text{and}$$
$$\int_x^\infty y^2 p_0(y)\,dy \leqslant x^{-1} \cdot \exp(-x^4)$$

Therefore,

$$\int_0^\infty v(x) q_0(x)\,dx$$
$$= (4\theta/\sigma^4)\left(\int_0^\infty \sigma^{-1} x^4 \cdot \exp\left\{ \left[(1-\theta)x^2 - \tfrac{1}{2}x^4 \right] / \sigma^2 \right\} - m_2/4 \right)$$
$$= -(\theta/3\sigma^4)(3m_2^2 - m_4) = -(-k_4)\theta/3\sigma^4 < 0$$

Therefore for $\mu = a \cdot q_0$

$$\begin{aligned} \mathscr{P} G_2 G_1^{-1} G_2 F_{f,\phi_0}(\mu) &= -\theta^2 \langle x, \mu \rangle f'(\langle \phi_0, \mu \rangle) \langle x, q_0 \rangle \langle \phi_0 / \langle \phi_0, q_0 \rangle, \mu \rangle^2 \\ &\quad \times \int v(x) q_0(x)\,dx \\ &= \theta^2 a^3 m_2^2 f'(a\langle \phi_0, q_0 \rangle) \times (-\theta k_4 / 3\sigma^4) \\ &= (-k_4)(\theta/12) a^3 f'(a\langle \phi_0, q_0 \rangle) \end{aligned} \tag{4.66}$$

and the proof of the lemma is complete. ■

Step IV. Weak Compactness of the Processes: $U_N(\cdot;\cdot)$. Let P^ϵ denote the probability measure on $C([0,\infty), M(R^1))$ associated with the process $\langle U_N(\cdot;\cdot), \phi_0 \rangle$ where $\epsilon = 1/N$. Let $U_N(0;dx) = \mu_\epsilon(dx)$ and assume that $\mu_\epsilon - p_0 \to q_0$ as $\epsilon \downarrow 0$. For $K > 0$, a stopping time τ_K is defined below in (4.71). Let $P^{\epsilon,K}$ denote the probability law of the process $\langle U_N(\cdot;\cdot), \phi_0 \rangle$ stopped at the stopping time τ_K.

Lemma 4.2.2.5. The probability measures $P^{\epsilon,K}$ are weakly compact in the sense of weak convergence of probability measures on $C([0,\infty), M_q^{\pm})$.

Proof. Let $F^\epsilon := F_{f,\phi_0} + \epsilon F_1$, where F_1 is defined by (4.41). Then

$$M_{F^\epsilon}(t) := F^\epsilon\big[U_{(1/\epsilon)}(t)\big] - F^\epsilon\big[U_{(1/\epsilon)}(0)\big] - \int_0^t G_\epsilon F^\epsilon\big[U_{(1/\epsilon)}(s)\big]\,ds \tag{4.67}$$

is a $P^{\epsilon,K}$ martingale. Then

$$\begin{aligned}M_{F^\epsilon}(t) &= F_{f,\phi_0}\big[U_{(1/\epsilon)}(t)\big] - F_{f,\phi_0}\big[U_{(1/\epsilon)}(0)\big]\\ &\quad + \epsilon F_1\big[U_{(1/\epsilon)}(t)\big] - \epsilon F_1\big[U_{(1/\epsilon)}(0)\big]\\ &\quad - \int_0^t \Big\{G_2F_1\big[U_{(1/\epsilon)}(s)\big] + G_3F_{f,\phi_0}\big[U_{(1/\epsilon)}(s)\big] + \epsilon G_3F_1\big[U_{(1/\epsilon)}(s)\big]\\ &\qquad + \epsilon G_4F_{f,\phi_0}\big[U_{(1/\epsilon)}(s)\big] + \epsilon^2G_4F_1\Big\}\,ds\end{aligned}$$

The increasing process associated with M_{F^ϵ} is given by

$$\langle M_{F^\epsilon}(t), M_{F^\epsilon}(t)\rangle = \int_0^t Q_{F^\epsilon}\big[U_{(1/\epsilon)}(s)\big]\,ds \tag{4.68}$$

where

$$Q_{F^\epsilon}(\mu) = G_\epsilon(F^\epsilon)^2 - 2F^\epsilon G_\epsilon F^\epsilon = \sigma^2\Big[(f'(\langle\phi_0,\mu\rangle))^2\langle(\partial\phi_0/\partial x)^2, p_0\rangle\Big]$$

By (4.61),

$$F_1(\mu) = -G_1^{-1}G_2F_{f,\phi_0}(\mu) = \theta\int_0^\infty f'(\langle T_t\phi_0,\mu\rangle)\langle T_t x,\mu\rangle\langle T_t(\partial\phi_0/\partial x),\mu\rangle\,dt$$

and $F_1(q_0) = 0$.

Since both $\langle U_{(1/\epsilon)}(\cdot),\phi_0\rangle$ and the limit $\langle Z,\phi_0\rangle$ are supported in $C[0,\infty)$, it suffices to show that the family $\{\langle U_{(1/\epsilon)}(\cdot),\phi_0\rangle : \epsilon > 0\}$ is relatively weakly compact as measures on $D[0,\infty)$, the space of right continuous functions having left limits. But to prove the latter it suffices to show that

$$\limsup_{\delta\downarrow 0}\lim_{\epsilon\downarrow 0}\sup_{|s-t|\leqslant\delta} E\Big[(\langle U_{(1/\epsilon)}(t),\phi_0\rangle - \langle U_{(1/\epsilon)}(s),\phi_0\rangle)^2\,|\,\mathscr{F}_s\Big] = 0 \tag{4.69}$$

where $\mathscr{F}_s$ denotes the σ-algebra $\sigma\{U_{(1/\epsilon)}(t) : 0 \leqslant t \leqslant s\}$. Taking $F_{f,\phi_0}(\mu) = \langle\phi_0,\mu\rangle$, we have

$$F_1(\mu) = \theta\int_0^\infty \langle T_t x,\mu\rangle\langle T_t(\partial\phi_0/\partial x),\mu\rangle\,dt, \quad\text{and}$$

$$\begin{aligned}&\langle\phi_0, U_{(1/\epsilon)}(t)\rangle - \langle\phi_0, U_{(1/\epsilon)}(s)\rangle\\ &\quad = -\epsilon\Big\{F_1\big[U_{(1/\epsilon)}(t)\big] - F_1\big[U_{(1/\epsilon)}(s)\big]\Big\}\\ &\qquad + \int_s^t\Big\{G_2F_1\big[U_{(1/\epsilon)}(u)\big] + \epsilon G_3F_1\big[U_{(1/\epsilon)}(u)\big] + \epsilon^2G_4F_1\big[U_{(1/\epsilon)}(u)\big]\Big\}\,du\\ &\qquad + M^\epsilon(t) - M^\epsilon(s)\end{aligned}$$

Hence,

$$\begin{aligned}
&\left[\langle \phi_0, U_{(1/\epsilon)}(t)\rangle - \langle \phi_0, U_{(1/\epsilon)}(s)\rangle\right]^2 \\
&\quad \leqslant 4\epsilon^2 \cdot \left\{F_1\left[U_{(1/\epsilon)}(t)\right] - F_1\left[U_{(1/\epsilon)}(s)\right]\right\}^2 \\
&\qquad + 4\left[\int_s^t \left\{G_2F_1\left[U_{(1/\epsilon)}(u)\right] du + \epsilon G_3F_1\left[U_{(1/\epsilon)}(u)\right]\right.\right. \\
&\qquad\qquad \left.\left. + \epsilon^2 G_4F_1\left[U_{(1/\epsilon)}(u)\right]\right\} du\right]^2 \\
&\qquad + 4\left[M^\epsilon(t) - M^\epsilon(u)\right]^2
\end{aligned}$$

Hence,

$$\begin{aligned}
&E\left(\left\{\langle \phi_0, U_{(1/\epsilon)}(t)\rangle - \langle \phi_0, U[_{(1/\epsilon)}(s)]\rangle\right\}^2 | \mathscr{F}_s\right) \\
&\quad \leqslant E\left\{4\epsilon^2\theta \int\int g(x,y)\left[U_{(1/\epsilon)}(t;dx)U_{(1/\epsilon)}(t;dy)\right.\right. \\
&\qquad\qquad \left. - U_{(1/\epsilon)}(s;dx)U_{(1/\epsilon)}(s;dy)\right] \\
&\qquad \left. + 4\sigma^3\left\langle(\partial\phi_0/\partial x)^2, p_0\right\rangle(t-s) + 4\left[\int_s^t H(U_{(1/\epsilon)}(u))\,du\right]^2 | \mathscr{F}_s\right\}
\end{aligned} \tag{4.70}$$

where

$$g(x,y) := \int_0^\infty T_t x \cdot T_t\left[u(y) - \int u(z)p_0(z)\,dz\right] dt$$

$u(\cdot)$ is defined as in (4.51) and

$$H(\mu) := G_2F_1(\mu) + \epsilon G_3F_1(\mu) + \epsilon^2 G_4F_1(\mu)$$

For $K > 0$, let

$$\tau_K := \min\left(K, \inf\left\{t : \max\left[\langle\phi_0, U(s)\rangle, \langle g(\cdot,\cdot), U(s)\times U(s)\rangle\right] \geqslant K\right\}\right) \tag{4.71}$$

Inequality (4.70) implies that the probability measures $P^{\epsilon,K}$ on $C[0,\infty)$ induced by the processes $\langle\phi_0, U_{(1/\epsilon)}(\cdot)\rangle$ stopped at τ_K are weakly compact and the proof of the lemma is complete. ■

Step V. Completion of the Proof of Theorem 4.2.2. By the weak compactness of the probability laws, $P^{\epsilon,K}$, it follows that there exist weak limits P^K with $P^K(Z(0) = a \cdot q_0) = 1$ where $\{Z(t) : t \geqslant 0\}$ denotes the canonical limit process. To complete the proof of the theorem it remains to prove that any such limit law must coincide with the probability law induced by the process $Z(\cdot)$ defined by (4.23) when stopped at τ_K.

For each $\epsilon > 0$ and function F_{f,ϕ_0} with $f \in C_0^2(R^1)$,

$$F^{\epsilon}[U_{(1/\epsilon)}(t)] - \int_0^{t \wedge \tau_K} G_{\epsilon} F^{\epsilon}[U_{(1/\epsilon)}(s)]\,ds \tag{4.72}$$

is a $P^{\epsilon,K}$-martingale. Therefore if Ψ_s is a bounded $\mathscr{F}_s$-measurable function,

$$E^{\epsilon,K}\left(\left\{F^{\epsilon}[U_{(1/\epsilon)}(t)] - F^{\epsilon}[U_{(1/\epsilon)}(s)] - \int_{s \wedge \tau_K}^{t \wedge \tau_K} G_{\epsilon} F^{\epsilon}[U_{(1/\epsilon)}(s)]\,ds\right\}\Psi_s\right) = 0 \tag{4.73}$$

But by (4.58)

$$G_{\epsilon} F^{\epsilon} = G_Z F_{f,\phi_0} + R(\epsilon) \tag{4.74}$$

where the remainder term $R[\epsilon, U_{(1/\epsilon)}(\cdot)] \to 0$ uniformly on $[0, \tau_K]$. It then follows from (4.73) and the weak convergence that for any weak limit of $P^{\epsilon,K}$,

$$E^K\left(\left\{F_{f,\phi_0}[Z(t \wedge \tau_K)] - F_{f,\phi_0}[Z(s \wedge \tau_K)] - \int_{s \wedge \tau_K}^{t \wedge \tau_K} G_Z F_{f,\phi_0}[Z(u)]\,du\right\} \cdot \Psi_s\right)$$
$$= 0 \tag{4.75}$$

The martingale problem associated with the operator G_Z has a unique solution and the resulting Markov process has no explosions. Using this fact together with (4.75) and the results of Stroock and Varadhan (Section 11.1)[54] we conclude that P^K coincides with the probability law associated with the solution of the G_Z-martingale problem when stopped at τ_K and that the probability measures $P^{\epsilon,K}$ converge weakly to P^K as $\epsilon \to 0$. Since $\tau_K \to \infty$, with probability 1, for the limit process $Z(\cdot)$, this also implies that P^{ϵ} converge weakly to P, the probability law of $Z(\cdot)$ in the sense of weak convergence of probability measures on $C([0,T], M_q^{\pm})$ for $T < \infty$. This completes the proof of Theorem 4.2.2. ■

Proof of Corollary 4.2.2. (a) Note that if $U_N(\infty; dx)$ has a weak limit as $N \to \infty$, then it is an equilibrium probability measure for $Z(\cdot;\cdot)$. However, since $Z(\cdot;\cdot)$ has a unique equilibrium distribution, $Z_{\infty}(dx)$, it follows that such a limit must coincide with $Z_{\infty}(dx)$. Hence it suffices to prove that the equilibrium random signed measures $U_N(\infty; dx)$ are weakly compact in the topology of weak convergence of probability distributions on $M^{\pm}(R^1)$. For this it suffices to show that for $g \in C_K(R^1)$, the family $\{\langle g, U_N(\infty)\rangle : N \geqslant 1\}$ is weakly compact. But if we denote the locations of the N particles at steady state by $X_1, \ldots, X_N$,

$$\langle g, U_N(\infty)\rangle = (N^{1/4}/N)\left\{\sum_{j=1}^{N} (g(X_j) - E[g(X_j)])\right\} \tag{4.76}$$

Now consider the random variables $\{[X_j, g(X_j)]: j = 1, \ldots, N\}$. The conditional law of $g(X_1), \ldots, g(X_N)$ given $\bar{X}_N$ under the probability law $p_N(\cdot)$ given by (4.12) is the same as that with respect to $\prod_{j=1}^{N}\rho(x_j)$. By (4.19), $N^{-1/2}\sum_{j=1}^{N}X_j$ is of order $O(N^{1/4})$ under the law $p_N(\cdot)$. Then Richter's multidimensional local limit theorem for large deviations (see Appendix C for a statement of this result, the limiting conditional density function for $\langle g, U_N(\infty)\rangle$ given $\bar{X}_N$ when $N^{1/2}\bar{X}_N$ if of order $O(N^{1/4})$ can be computed. In particular it yields a limiting conditional limiting density

$$\lim_{N\to\infty} p_N(u\,|\,y) = p(u\,|\,y) \tag{4.77}$$

where $p_N(\cdot\,|\,\cdot)$ is the conditional density for

$$N^{-1/2}\sum_{j=1}^{N}\{g(X_j) - E_N[g(X_j)]\}$$

conditioned on

$$N^{-1/2}\sum_{j=1}^{N}X_j = N^{1/4}y$$

provided that $\int xg(x)p_0(x)\}\,dx = 0$, $g(\cdot)$ is bounded and $\int g^2(x)p_0(x)\}\,dx > 0$. Referring to Appendix C, only the terms involving Q_3 and Q_4 contribute to the limiting conditional distribution of $\langle g, U_N(\infty)\rangle$ and the latter distribution is Gaussian. The existence of a limiting conditional density together with the weak compactness of $N^{1/4}\cdot\bar{X}_N$ yields the required weak compactness of the family $\langle g, U_N(\infty)\rangle$ and the proof of (a) is complete. ■

(b) In order to prove (b) it suffices to show that all moments of the empirical mean converge, that is,

$$E\big(\langle x, U_N(\infty)\rangle^k\big) \to E\big(\langle x, Z_\infty\rangle^k\big) \qquad \text{as} \quad N\to\infty \tag{4.78}$$

In view of the weak convergence established in part (a), it suffices to establish uniform integrability, that is,

$$\limsup_{N\to\infty} E\Big[\big(N^{1/4}\cdot\bar{X}_N\big)^k\Big] \leqslant K_k < \infty \tag{4.79}$$

Using the results of (4.13), (4.14) it follows that for $0 < s_0 < 1$,

$$G(s) \geqslant c_1 s^2 \qquad \text{for} \quad |s| \geqslant s_0 \tag{4.80}$$

$$G(s) = c_2 s^4 + O(s^6) \qquad \text{for} \quad |s| \leqslant s_0, \quad \text{with} \quad c_2 > 0 \tag{4.81}$$

where c_1, c_3 are constants and $O(s^6)$ is bounded on $|s| \leqslant s_0$. Then (4.18) together with (4.80) and (4.81) imply that

$$\begin{aligned} N^{1/4}\cdot \bar{X}_N + W/N^{1/4} &\sim Z^{-1}\cdot \exp\left[-c_2 s^4 + O(s^6)/N^{1/2}\right] && \text{for } |s| \leqslant N^{1/4}s_0 \\ &\sim Z^{-1}\cdot \exp(-N^{1/2}s^2) && \text{for } |s| \geqslant N^{1/4}s_0 \end{aligned} \tag{4.82}$$

Therefore,

$$E\left[\left(N^{1/4}\cdot \bar{X}_N\right)^k\right] \leqslant c_3\left[\int_{-\infty}^{\infty} x^k \exp(-c_2x^4)\,dx\right] + c_4\int_{-\infty}^{\infty} x^k\cdot \exp(-N^{1/2}x^2)\,dx \tag{4.83}$$

The second term on the right-hand side of (4.83) goes to zero as $N \to \infty$. Therefore for each positive integer k,

$$\limsup_{N\to\infty} E\left[\left(N^{1/4}\cdot \bar{X}_N\right)^k\right] \leqslant K_k < \infty \tag{4.84}$$

and the proof of (4.79) is complete. This completes the proof of Corollary 4.2.2. ■

APPENDIX A: PROOF OF THEOREM 2.4.1.

A.1. Step I: A Priori Bounds

Let $x(\cdot)$ denote the solution of the first-order linear equation:

$$dx(t)/dt = -f(t)x(t) + g(t), \qquad x(0) = x_0$$

Multiplying by the integrating factor $\exp[\int_0^t f(s)\,ds]$ we obtain

$$x(t)\exp\left[\int_0^t f(s)\,ds\right] - x_0 = \int_0^t g(s)\exp\left[\int_0^s f(u)\,du\right]ds \tag{A1}$$

We apply this to Eq. (2.23) with

$$f(t) = \left[(\theta - 1) + y^2(t)\right]$$
$$g(t) = \sigma w'(t) + \theta m_1(t)$$

where $w'(t)$ denotes the generalized derivative of $w(t)$. Since $\exp[\int_0^t f(s)\,ds]$ is differentiable, this computation proceeds by the usual rules of calculus. This yields

$$y(t)\exp\left[\int_0^t f(s)\,ds\right] = x_0 + \theta\int_0^t m_1(s)\exp\left[\int_0^s f(u)\,du\right]ds + \sigma\int_0^t \exp\left[\int_0^s f(u)\,du\right]dw(s)$$

Integrating by parts we obtain

$$y(t) = x_0\exp\left[-\int_0^t f(s)\,ds\right] + \theta\int_0^t m_1(s)\exp\left[-\int_s^t f(u)\,du\right]ds + \sigma w(t)$$

$$-\sigma e^{(\theta-1)t}\exp\left[-\int_0^t y^2(s)\,ds\right]\int_0^t\left\{w(s)\left[(\theta-1)+y^2(s)\right]e^{(1-\theta)s}\right.$$

$$\left.\times\exp\left[\int_0^s y^2(u)\,du\right]\right\}ds \qquad \text{(A2)}$$

Hence

$$E|y(t)| \leqslant |x_0|e^{(1-\theta)t} + \theta\int_0^t |m_1(s)|e^{(1-\theta)(t-s)}\,ds + \sigma E|w(t)|$$

$$+\sigma E\left\{\sup_{0\leqslant s\leqslant t}|w(s)|\right\}(1+e^{(\theta-1)t})$$

By the submartingale inequality,

$$E\left\{\sup_{0\leqslant s\leqslant t}|w(s)|\right\} \leqslant E\left\{\sup_{0\leqslant s\leqslant t}|w(s)|^2\right\}^{1/2} \leqslant 2t^{1/2}$$

Letting $m^+(t) \equiv E|y(t)|$, we obtain

$$m^+(t) \leqslant |x_0|e^{(1-\theta)t} + 2\sigma t^{1/2} + \theta\int_0^t m^+(t)e^{(1-\theta)(t-s)}\,ds$$

$$+2\sigma t^{1/2}(1+e^{(\theta-1)t})$$

Therefore, for $0 \leqslant t \leqslant T$, $\theta \neq 0$,

$$m^+(t) \leqslant (K|x_0| + 2\sigma T^{1/2})\exp(\theta + \theta e^{t(1-\theta)}/|1-\theta|),$$

$$K = \max(1, e^{(1-\theta)T}) \qquad \text{(A3)}$$

This implies that $m_1(t) \leqslant m^+(t)$ is bounded on $[0, T]$. Together with (A2) this also implies the continuity of $m_1(t)$ and the nonexistence of explosions for the equation (2.22).

Step II: Picard Iteration

Let $y_1(t)$ denote the unique strong solution of

$$dy_1(t) = \left[-y_1^3(t) + y_1(t)\right]dt + \sigma\,dw(t) - \theta y_1(t)\,dt + \theta y(0)$$

The fact that this equation (and those below) have unique strong solutions is established by using the Cameron–Martin–Girsanov change of probability formula together with the fact that the solution is nonexplosive (cf. Ikeda–Watanabe[29] or McKean[42]).

We now set up the iteration scheme: for $n \geqslant 1$,

$$y_{n+1}(t) - y_{n+1}(0) = \int_0^t \left[-y_{n+1}^3(s) + (1-\theta) y_{n+1}(s) \right] ds + \sigma w(t)$$
$$+ \theta \int_0^t m_1^{(n)}(s)\, ds \tag{A4}$$
$$m_1^{(n)}(t) \equiv E\left[y_n(t) \right]$$

This yields

$$y_{n+1}(t) - y_n(t) = \int_0^t (1-\theta)\left[y_{n+1}(s) - y_n(s) \right] ds - \int_0^t \left[y_{n+1}^3(s) - y_n^3(s) \right] ds$$
$$+ \theta \int_0^t \left[m_1^{(n)}(s) - m_1^{(n-1)}(s) \right] ds$$

Since $(y^3 - x^3) = (y-x)(y^2 + xy + x^2) = \frac{1}{2}(y-x)[(x+y)^2 + x^2 + y^2]$,

$$y_{n+1}(t) - y_n(t) = \int_0^t (1-\theta)\left[y_{n+1}(s) - y_n(s) \right] ds$$
$$- \int_0^t \left[y_{n+1}(s) - y_n(s) \right] f(s)\, ds$$
$$+ \theta \int_0^t \left[m_1^{(n)}(s) - m_1^{(n-1)}(s) \right] ds$$
$$y_{n+1}(0) - y_n(0) = 0$$

where $f(s) = \frac{1}{2}\{[y_{n+1}(s) + y_n(s)]^2 + y_{n+1}(s)^2 + y_n(s)^2\} \geqslant 0$. Applying (A1) we obtain

$$y_{n+1}(t) - y_n(t) = \theta \int_0^t e^{(t-s)(\theta-1)} \exp\left[-\int_s^t f(u)\, du \right]$$
$$\cdot \left[m_1^{(n)}(s) - m_1^{(n-1)}(s) \right] ds$$

Hence for $0 \leqslant t \leqslant T$,

$$|m_1^{(n+1)}(t) - m_1^{(n)}(t)| \leqslant K \int_0^t \left| \left[m_1^{(n)}(s) - m_1^{(n-1)}(s) \right] \right| ds \tag{A5}$$

where $K = \theta \max(1, e^{T(\theta-1)})$. Thus for $0 \leqslant t \leqslant T$, $n \geqslant 1$,

$$|m_1^{(n+1)}(t) - m_1^{(n)}(t)| \leqslant K^n T^n / n!$$

This implies that $\{m_1^{(n)}(t) : 0 \leqslant t \leqslant T\}$ is Cauchy and the limit $m_1(t)$ is continuous and bounded on $[0, T]$. Let $y(\cdot)$ denote the unique strong solution of the equation

$$dy(t) = \left[-y^3(t) + (1-\theta) y(t) \right] dt + \sigma\, dw(t) + \theta m_1(t)\, dt$$

By an argument similar to the above we can verify that

$$|E[y(t)] - m_1^{(n)}(t)| \leqslant K\int_0^t |m_1(s) - m_1^{(n)}(s)|\, ds$$

Then it follows that $E[y(t)] = m_1(t)$ and the existence of a strong solution to (2.22) is established.

Step III: Uniqueness

Let $y_1(\cdot)$ and $y_2(\cdot)$ denote two solutions of (2.22) and let $m_1^{(1)}(t) = E[y_1(t)]$ and $m_1^{(2)}(t) = E[y_2(t)]$. Then as above we can obtain

$$|m_1^{(2)}(t) - m_1^{(1)}(t)| \leqslant K\int_0^t |m_1^{(2)}(s) - m_1^{(1)}(s)|\, ds$$

Hence by Gronwall's inequality, $m_1^{(2)}(t) = m_1^{(1)}(t)$, thus establishing the uniqueness of the solution to (2.22).

A.2. Existence

Let $y(\cdot)$ denote the solution of (2.22). If $\phi \in \mathscr{S}$, Itô's stochastic chain rule yields that

$$\phi(y(t)) - \phi(y(0)) - \int_0^t \Big[\big(-y^3(s) + y(s)(1-\theta)\big)\phi'(y(s)) + \tfrac{1}{2}\sigma^2\phi''(y(s)) + \theta m_1(t)\phi'(y(s))\Big]\, ds \tag{A6}$$

is a martingale. Let $p(t; dx)$ denote the probability law of $y(t)$. Then the martingale condition (A6) implies that $p(\cdot\,;\cdot)$ is a probability-measure-valued solution of the nonlinear equation (2.23).

Uniqueness. Suppose that $p^*(\cdot\,;\cdot)$ is a second solution of (2.23). Let $m_1^*(t) := \int x p^*(t; dx)$. Let $y^*(\cdot)$ denote the unique strong solution of the stochastic differential equation:

$$dy^*(t) = \big[y^*(t)^3 + (1-\theta)y^*(t)\big]\, dt + \sigma\, dw(t) + \theta m_1^*(t)\, dt$$

By the uniqueness established in Step III, $m_1^*(t) = m_1(t)$. Therefore $p^*(\cdot\,;\cdot)$ is a solution of the linear equation:

$$\partial p^*(t;\cdot)/\partial t = \tfrac{1}{2}\sigma^2\,\partial^2 p^*(t;\cdot)/\partial x^2 - \partial/\partial x\big[(1-\theta)x - x^3\big]p^*(t;\cdot) - \theta m_1(t)\,\partial p^*(t;\cdot)/\partial x \tag{A7}$$

Provided that $m_1(\cdot)$ is C^∞, it can be proved (cf. McKean[42]) that (A7) has a unique solution and that $p^*(t; dx) = p(t; x)\, dx$, where $p(t; x)$ is in $C([0, \infty) \times R^1)$. To verify that $m_1(\cdot)$ is C^∞, first note that we can obtain a estimate $P(|y(t)| \geqslant x) \leqslant c_1 \cdot \exp(-c_2 x^2)$, where c_1, c_2 are constants, for $0 \leqslant t \leqslant T$ and $|x| \geqslant c_3$ (e.g., by comparison with an appropriate reflecting Brownian motion). This estimate allows us to extend the validity of (A6) to polynomials. Repeated application of (A6) then leads to an integral expression for $d^k m_1(t)/dt^k$ for each k and this completes the proof that $m_1(\cdot) \in C^\infty$. This completes the proof of (b). ■

APPENDIX B: PROOF OF THEOREM 2.5.1

Step I. Weak Compactness

In order to prove that the processes are compact in the weak topology on $C([0,\infty), M_1(R^1))$ it suffices to prove that

(B-i) $\sup_N \sup_{0 \leqslant t \leqslant T} E(\langle X_N(t), |x| \rangle) < \infty$, and

(B-ii) for each $\phi \in C_K(R^1)$, the space of functions with compact support, $\langle X_N(\cdot), \phi \rangle$ are weakly compact in the topology of weak convergence of probability measures on $C[0,\infty)$.

By a calculation similar to that of Step I of Appendix A we obtain for $j = 1, \ldots, N$

$$\begin{aligned} x_j(t) = {} & x_j(s) \cdot \exp\left[-\int_s^t x_j^2(u)\,du\right] \exp\left[(1-\theta)(t-s)\right] + \sigma w_j(t) \\ & + \theta \int_s^t \bar{x}(r) \exp\left[-\int_r^t x_j^2(u)\,du\right] \exp\left[(1-\theta)(t-r)\right] dr \\ & - \sigma \exp\left[(\theta-1)t\right] \exp\left[-\int_s^t x_j^2(u)\,du\right] \\ & \cdot \int_s^t \left\{ w_j(u)\left[(\theta-1) + x_j^2(u)\right] \exp\left[(1-\theta)u\right] \exp\left[\int_s^u x_j^2(v)\,dv\right]\right\} du \end{aligned} \tag{B1}$$

Let $z(t) := \langle X_N(t), |x| \rangle := N^{-1} \cdot \sum_{j=1}^{N} |x_j(t)|$. Then from (B1) we obtain

$$\begin{aligned} z(t) \leqslant {} & z(0) \cdot \exp\left[(1-\theta)(t-s)\right] + \int_s^t \theta z(r) \cdot \exp\left[(1-\theta)(t-r)\right] dr \\ & + \sigma |\bar{w}(t)| + \sigma N^{-1} \cdot \sum_{j=1}^{N} \sup_{s \leqslant u \leqslant t} |w_j(u)| \cdot \left\{1 + \exp\left[(\theta-1)(t-s)\right]\right\} \end{aligned} \tag{B2}$$

Taking expectations in (B2) and using Gronwall's inequality leads to (B-i). Similarly,

$$\begin{aligned}|x_j(t) - x_j(s)| \leqslant \theta\int_s^t |\bar{x}(r)|\cdot \exp\big[(1-\theta)(t-r)\big]\,dr + \sigma|w_j(t) - w_j(s)| \\ + \sigma \sup_{s\leqslant u\leqslant t} |w_j(u)|\cdot\big(1+\exp\big[(\theta-1)(t-s)\big]\big) \\ + |x_j(s)|\,\big|\{\exp\big[(1-\theta)(t-s)\big]-1\}\big| \\ + |x_j(s)|\sup_{s\leqslant u\leqslant t} x_j^2(u)\cdot\exp\big[(1-\theta)(t-s)\big]\cdot(t-s)\end{aligned} \tag{B3}$$

Then

$$\begin{aligned}\langle X_N(t),\phi\rangle - \langle X_N(s),\phi\rangle &= N^{-1}\cdot\sum_{j=1}^N\big[\phi(x_j(t)) - \phi(x_j(s))\big] \\ &\leqslant \sup_x|\phi'(x)|\cdot N^{-1}\cdot\sum_{j=1}^N|x_j(t)-x_j(s)| \qquad \text{(B4)}\end{aligned}$$

Using (B1) an inequality similar to (B2) can be obtained for $N^{-1}\cdot\sum_{j=1}^N x_j(\cdot)^2$. This estimate together with (B3) and (B4) implies that $\{\langle X_N(\cdot),\phi\rangle : N = 1,2,3,\ldots\}$ are weakly compact in $C([0,T])$ for every $T<\infty$ and $\phi\in C_K(R^1)$. This completes the proof of the weak compactness. ■

Step II. Identification of the Limit

Let P_N denote the probability law of the probability-measure-valued process X_N. In view of the weak compactness there exist convergent subsequences $P_{N_k}\twoheadrightarrow P$. It remains to prove that P is uniquely characterized as the solution of a martingale problem having a unique solution, namely, the probability law of the deterministic process $\{Y(t): t\geqslant 0\}$.

Let $f\in C_K(R^n)$. Then by (2.13),

$$\begin{aligned}G_NF_f(\mu) = &\sum_{j=1}^n\int_{R^n}\big\{\tfrac{1}{2}\sigma^2\partial^2/\partial x_j^2 + \big[(1-\theta)x_j - x_j^3\big]\partial/\partial x_j\big\} \\ &\times f(x_1,\ldots,x_n)\,\mu_n(d\mathbf{x}) \\ &+(\sigma^2/2N)\sum_{j=1}^n\sum_{k\neq j}\int_{R^n}\big\{\partial^2/\partial x_j\,\partial x_k\big[f(x_1,\ldots,x_n)\big]\big\} \\ &\times\delta(x_j-x_k)\,\mu_{n-1}(d\mathbf{x}) \\ &+\theta\sum_{j=1}^n\int_{R^{n+1}}\big[x_{n+1}(\partial/\partial x_j f(x_1,\ldots,x_n))\big]\,\mu_{n+1}(d\mathbf{x})\end{aligned}$$

Then

$$\lim_{N\to\infty} G_N F_f(\mu) = \sum_{j=1}^{n} \int_{R^n} \left\{ \tfrac{1}{2}\sigma^2 \partial^2/\partial x_j^2 + \left[x_j(1-\theta) - x_j^3 \right] \partial/\partial x_j \right\}$$
$$\times f(x_1, \ldots, x_n)\, \mu_n(d\mathbf{x})$$
$$+\theta \sum_{j=1}^{n} \int_{R^{n+1}} \left[x_{n+1} \cdot \partial/\partial x_j\, f(x_1, \ldots, x_n) \right] \mu_{n+1}(d\mathbf{x})$$
$$= GF_f(\mu) \tag{B5}$$

Therefore the limit measure P on Ω must be a solution of the martingale problem associated with the martingale problem (G, D_1). But according to Theorem 2.4.1, this martingale problem has a unique solution. This completes the proof of Theorem 2.5.1. ■

APPENDIX C: RICHTER'S THEOREM ON LARGE DEVIATONS

In this appendix we state the multidimensional local limit theorem for large deviations due to Richter[48] (see also Petrov[47]).

Let $X^{(1)}, X^{(2)}, X^{(3)}, \ldots$ denote independent identically distributed m-dimensional random vectors with distribution function $V(dx_1, \ldots, dx_m)$. Assume that all mixed second moments exist:

$$\sigma^{jk} := E\left\{ \left[X_j^{(1)} - E\left(X_j^{(1)}\right) \right] \left[X_k^{(1)} - E\left(X_k^{(1)}\right) \right] \right\}$$
$$\Sigma := \|\sigma^{jk}\|, \qquad \text{an } m \times m \text{ matrix}$$

Assume that $D = \det \Sigma > 0$. Let $\xi = (x_1, \ldots, x_m)$, $\mathbf{v} = (v_1, \ldots, v_m)$ and $\langle \mathbf{v}, \xi \rangle := \sum_{j=1}^{n} x_j v_j$.

Theorem. Let $E(X^{(1)}) = \mathbf{0}$ and let $Z_N := N^{-1/2} \cdot \sum_{j=1}^{N} X^{(j)}$. Assume that

(a) for $N > N_0$, Z_N has a bounded density $p_{Z_N}(\xi)$, and

(b) there exists a positive α such that for all $\mathbf{v} \in R^m$ with $|v_j| < \alpha$, the integral $\int_{R^m} \exp(\langle \mathbf{v}, \xi \rangle) V(d\xi)$ converges.

Then for all sufficiently large N and $|x_j| > 1$, $|x_j| = o(N^{1/2})$,

$$p_{Z_N}(\xi)/\phi(\xi) = \exp\left\{ N \cdot \sum_{k=3}^{\infty} Q_k(\xi/N^{1/2}) \left[1 + O(|\xi|/N^{1/2}) \right] \right\} \tag{C1}$$

In (C1) $\phi(\xi)$ denotes the density of the m-dimensional Gaussian distribution

$$\phi(\xi) = \exp\left(-\tfrac{1}{2} \xi \cdot \Sigma^{-1} \cdot \xi^T \right) / (2\pi)^{(1/2)m} \cdot D^{1/2}$$

$Q_k(\mathbf{v})$ is a multilinear form:

$$Q_k(\mathbf{v}) = \sum_{j_1, \ldots, j_k = 1}^{m} a_{j_1, \ldots, j_k} v_{j_1} v_{j_2} \cdots v_{j_k}$$

where the coefficients $a_{j_1,\ldots,j_k}$ are computed from the joint cumulants of order less than or equal to k of the original distribution $V(d\xi)$.

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